Please note that these solutions are only suggestions; different answers and proofs are also always possible.

• Section 5.1: Counterexamples

1. Exercise 5.6: Let $a, b \in \mathbb{Z}$. Disprove the statement:

If $ab$ and $(a + b)^2$ are of opposite parity, then $a^2b^2$ and $a + ab + b$ are of opposite parity.

Solution: Let $a = b = 1$, then $ab = 1$ and $(a + b)^2 = 4$ are of opposite parity, but $a^2b^2 = 1$ and $a + ab + b = 3$ are of same parity. (In general, the statement is false whenever $a$ and $b$ are both odd.)

• Section 5.2: Proof by Contradiction

2. Exercise 5.16: Prove that $\sqrt{3}$ is irrational. [Hint: First prove for an integer $a$ that $3 \mid a^2$ if and only if $3 \mid a$. Recall that every integer can be written as $3q$, $3q + 1$, or $3q + 2$ for some integer $q$.]

Solution: To first verify the hint, let $a$ be an integer. Then either $a \equiv 0$, $a \equiv 1$, or $a \equiv 2$ (mod 3) and thus $a^2 \equiv 0$, $a^2 \equiv 1$, or $a^2 \equiv 4 \equiv 1$ (mod 3), respectively. This shows that $3 \mid a^2$ if and only if $3 \mid a$.

Proof. By contradiction, now suppose that $\sqrt{3}$ is rational so that there are two integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $\sqrt{3} = p/q$. Without loss of generality, we can assume that $p$ and $q$ are co-prime and have no common divisors (otherwise reduce both $p$ and $q$ until they become co-prime). If we square the former equality and rearrange, we find that $3q^2 = p^2$ which implies that $3$ divides $p^2$, and equivalently, that $3$ divides $p$. Hence, we can write $p = 3r$ for some integer $r$ and substitute to find $3q^2 = (3r)^2 = 9r^2$ so that $q^2 = 3r^2$. This shows that $3$ also divides $q^2$ and $q$ in contradiction to our assumption that $p$ and $q$ are co-prime. So our starting assumption must have been wrong, and $\sqrt{3}$ must be irrational.

• Section 5.3: A Review of Three Proof Techniques

3. Exercise 5.26: Let $x$ be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$ by

(a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.

Solution: The proofs use that we can multiply inequalities by positive integers and some basic bounds.

Direct Proof. Let $x$ be a positive real number so that $x - \frac{2}{x} > 1$. Because $x$ is positive, we can multiply the inequality by $x$ and find that $x^2 - 2 > x$, or equivalently, that $x^2 - x - 2 = (x-2)(x+1) > 0$. Because $x+1 > 1$ whenever $x > 0$, now we can divide this inequality by $x+1$ and find that $x - 2 > 0$, or equivalently, that $x > 2$. [You could also multiply the initial inequality directly by $x/(x+1) > 0$.]

Proof by Contrapositive. Let $x$ be a positive real number and, by contrapositive, suppose that $x \leq 2$. Then $\frac{2}{x} \geq 1$ (note that $x > 0$ by our general assumption) and thus $x - \frac{2}{x} \leq 2 - 1 = 1$. [Be careful with the inequality signs: an upper bound on a variable in the denominator gives a lower bound for the fraction, and a lower bound for a (positive) fraction gives an upper bound for the negated fraction.]

Proof by Contradiction. Let $x$ be a positive real number, let $x - \frac{2}{x} > 1$ and, by contradiction, suppose that $x \leq 2$. Then continue as in the direct proof or the proof by contradiction, and add the concluding phrase “in contradiction to our initial assumption”. [Personally, I usually try to avoid proofs by contradiction, and especially if it is possible to do a shorter direct proof or proof by contrapositive.]

• Section 5.4: Existence Proofs

4. Exercise 5.32: Show that there exist no nonzero real numbers $a$ and $b$ such that

$$\sqrt{a^2 + b^2} = \sqrt{a^3 + b^3}.$$
Solution: Take both sides to the sixth power, rearrange and collect terms, and simplify:
\[(a^2 + b^2)^3 = (a^3 + b^3)^2\]
\[a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = a^6 + 2a^3b^3 + b^6\]
\[3a^4b^2 - 2a^3b^3 + 3a^2b^4 = 0\]
\[a^2b^2(3a^2 - 2ab + 3b^2) = 0.\]

Now observe that \(a^2b^2 = 0\) if and only if \(a = 0\) or \(b = 0\), and \(3a^2 - 2ab + 3b^2 = 2a^2 + (a-b)^2 + 2b^2 = 0\) if and only if \(a = 0\) and \(b = 0\). Together, this shows that at least one of the two numbers \(a\) or \(b\) must be zero, and because of the cubes/third roots in the original equation, it follows that the other number must be nonnegative (because \(\sqrt[3]{a^2} = |a|\) is always nonnegative but \(\sqrt[3]{a^2} = a = |a|\) if and only if \(a \geq 0\)).

Please note that it is not correct to say that \(a\) lost a point if you did), because either \(a\) or \(b\) could also be positive as long the other number is zero.

• Section 5.5: Disproving Existence Statements

5. Exercise 5.36: Disprove the statement: There is a real number \(x\) such that \(x^6 + x^4 + 1 = 2x^2\).

\[\text{[This problem has a very elegant solution that is not trivial, however. If you cannot find it, then}
\text{try to find an easier solution under the additional assumption that the real number}\ x \text{is an integer.]}\]

Solution: We can rearrange the given equation to \(x^6 + x^4 - 2x^2 + 1 = x^6 + (x^2 - 1)^2 = 0\), where both terms are nonnegative and zero for different values of \(x\) so that there can be no real solution. If we assume that \(x\) is an integer, then a completely different proof is possible: In this case, we can observe (or prove) that the left-hand-side term \(x^6 + x^4 + 1\) is always odd regardless on whether \(x\) is even or odd, whereas the right-hand-side term \(2x^2\) is always even. Hence, there can be no integer solution.

• Additional Exercises for Chapter 5

6. Exercise 5.48: Let \(a_1, a_2, \ldots, a_r\) be odd integers where \(a_i > 1\) for \(i = 1, 2, \ldots, r\). Prove that if \(n = a_1a_2\cdots a_r + 2\), then \(a_i \nmid n\) for each integer \(i\) \((1 \leq i \leq r)\). [Note that this problem is very similar to the key idea in Euclid’s existence proof that there are infinitely many prime numbers.]

Solution: We use a proof by contradiction which turns out quite convenient for this type of argument.

Proof. Let \(a_i > 1\) be odd integers for \(i = 1, 2, \ldots, r\), define \(n = a_1a_2\cdots a_r + 2\) and, by contradiction, suppose that \(a_i \mid n\) for some integer \(i\). By definition of \(n\), this means that \(a_i \mid 2\) also, and thus \(a_i = 1\) or \(a_i = 2\) (which is even). This yields a contradiction because by definition, \(a_i > 1\) is an odd integer.

• Section 7.2: Revisiting Quantified Statements

7. Exercise 7.12:

(a) Express the following quantified statement in symbols:

\[\text{For every even integer}\ a \text{and odd integer}\ b, \text{there exists a rational number}\ c \text{such that either}\ a < c < b \text{or} b < c < a.\]

(b) Prove that the statement in (a) is true.

Solution: For (a), we can write \(\forall (a, b) \in \mathbb{Z} \times \mathbb{Z} : \exists c \in \mathbb{Q} : (2a < c < 2b + 1) \lor (2b + 1 < c < 2a)\).

Note that rather than using \(a\) and \(b\) for the even and odd integers itself, we used \(a\) and \(b\) as integers to produce such even or odd integer. Our proof for (b) assumes a similar level of mathematical maturity.

Proof. Let \(a\) be any even integer, \(b\) be any odd integer, and define \(c = (a + b)/2\). It is clear that \(c\) is rational. Moreover, because \(a \neq b\), either \(a\) is larger than \(b\), or \(b\) is larger than \(a\). Without loss of generality, suppose that \(a < b\) so that \(c = (a+b)/2 > (a+a)/2 = a\) and \(c = (a+b)/2 < (b+b)/2 = b\).

After grading your assignments, I learned the following, very elegant solution for (a) from some of you: Let \(E := \{a \in \mathbb{Z} : a = 2k\text{ for some integer } k \in \mathbb{Z}\}\) and \(O := \{b \in \mathbb{Z} : b = 2k+1\text{ for some integer } k \in \mathbb{Z}\}\) be the set of all even or odd integers, respectively. Then the statement can be written nicely in symbols:

\[\forall (a, b) \in E \times O : \exists c \in \mathbb{Q} : (a < c < b) \lor (b < c < a).\]
8. Exercise 7.14:
(a) Express the following quantified statement in symbols:
There exist odd integers \( a, b, \) and \( c \) such that \( a + b + c = 1 \).
(b) Prove that the statement in (a) is true.

Solution: Similar to Exercise 7.12, for (a) we can write \( \exists (a, b, c) \in \mathbb{Z}^3 \): \( (2a+1) + (2b+1) + (2c+1) = 1 \).
For (b), we give a simple example: let \( a = 1 \) and \( b = c = -1 \) (so the three integers are 3, -1, and -1).
Again much nicer, using \( O \) as the set of odd integers we could also write \( \exists (a, b, c) \in O^3 \): \( a + b + c = 1 \).

9. Exercise 7.62 (Prove or disprove): There exist three distinct integer \( a, b, \) and \( c \) such that \( a^b = b^c \).

Solution: This statement is true and can be proven by a simple example: let \( a = 8, b = 2, \) and \( c = 6, \) then \( a^b = b^c = 8^2 = 2^6 = 64 \). In fact, there are infinitely many examples: let \( b \) and \( k \) be any two integers such that \( b, a = b^k, \) and \( c = kb \) are pairwise distinct, and observe that \( a^b = (b^k)^b = b^{kb} = b^c \).
Here are some of your examples: \( 1^2 = 2^0 \) for \( (a, b, c) = (1, 2, 0) \); \( 1^{-1} = (-1)^2 \) for \( (a, b, c) = (1, -1, 2) \).

10. Exercise 7.70:
(a) Prove or disprove: There exist two distinct positive integers whose sum exceeds their product.
(b) Your solution to (a) should suggest another problem to you. State and solve this new problem.

Solution: The statement in (a) says that there exist two positive integers \( a \) and \( b \) such that \( a \neq b \) and \( ab < a + b \). This is actually true: let \( a = 1 \) and \( b > 1 \) be any other positive integer, then \( a = 1 \neq b \) and \( ab = b < b + 1 \). For (b), this suggests the natural question whether there exist two integers greater than 1 (either distinct or not) whose sum exceeds their product (note that for \( a = b = 2, \) their sum equals but does not exceed their product). It turns out that this question can be negated: wlog, let \( 2 \leq a \leq b \), then \( a + b \leq 2b\leq ab \) which shows that that product is never smaller than their sum.

I also liked the following proof that some of you gave, which does not need the “wlog” assumption that one number is at least as large as the other: let \( a \geq 2 \) and \( b \geq 2 \) so that \( a - 1 \geq 1 \) and \( b - 1 \geq 1 \) and thus \( 1 \leq (a-1)(b-1) = ab - a - b + 1 \) which is equivalent to \( a + b \leq ab \). This is indeed very elegant!

Please let me know if you have any questions, comments, corrections, or remarks.