Math 3000
Cantor's Diagonal Argument

Theorem
The set of real numbers $(0, 1)$ is uncountable.

Proof
Assume that the real numbers in the set $(0, 1)$ are countable. This means that these real numbers can be written in order as $\{x^{(1)}, x^{(2)}, x^{(3)}, \ldots, \}$. Each of these numbers can be written using a binary expansion. For notation,

$$x^{(k)} = 0.x_1^{(k)} x_2^{(k)} x_3^{(k)} x_4^{(k)} \cdots \qquad \text{where } x_j^{(k)} \in \{0, 1\}$$

Since the real numbers in $(0, 1)$ are countable we can write

$$\begin{array}{cccccc}
x^{(1)} = & x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & \cdots \\
x^{(2)} = & x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & \cdots \\
x^{(3)} = & x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & \cdots \\
x^{(4)} = & x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Now construct the number y where $y = 0.y_1 y_2 y_3 y_4 \cdots$. Choose $y_1 \neq x_1^{(1)}$, $y_2 \neq x_2^{(2)}, \ldots, y_j \neq x_j^{(j)}, \ldots$. This number is NOT in the original list $\{x^{(1)}, x^{(2)}, x^{(3)}, \ldots, \}$. This means that we have assumed that we have a countable list, and we have constructed a number that is not in the list. Therefore, we arrive at a contradiction that the real numbers are countable. QED

An example of such a construction is:

$$\begin{array}{cccccccccccc}
x^{(1)} = & \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
x^{(2)} = & 0 & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
x^{(3)} = & 1 & 0 & \boxed{1} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\
x^{(4)} = & 0 & 1 & 0 & \boxed{1} & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
x^{(5)} = & 0 & 1 & 1 & 0 & \boxed{1} & 1 & 0 & 1 & 1 & 0 & \cdots \\
x^{(6)} = & 1 & 0 & 1 & 0 & 0 & \boxed{1} & 0 & 1 & 0 & 1 & \cdots \\
x^{(7)} = & 1 & 0 & 1 & 0 & 0 & 1 & \boxed{0} & 0 & 1 & 0 & \cdots \\
x^{(8)} = & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \boxed{0} & 1 & 0 & \cdots \\
x^{(9)} = & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \boxed{0} & 1 & \cdots \\
x^{(10)} = & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \boxed{1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\hline
x_M \neq & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\hline
\end{array}$$