Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs

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Abstract

A rainbow matching in an edge-colored graph is a matching in which all the edges have distinct colors. Wang asked if there is a function $f(\delta)$ such that a properly edge-colored graph $G$ with minimum degree $\delta$ and order at least $f(\delta)$ must have a rainbow matching of size $\delta$. We answer this question in the affirmative; an extremal approach yields that $f(\delta) = 13\delta/3 - 2$ suffices. Furthermore, we give an $O(\delta(G)|V(G)|^2)$-time algorithm that generates such a matching in a properly edge-colored graph of order at least $6.5\delta$.

Keywords: Rainbow matching, properly edge-colored graphs

1 Introduction

All graphs under consideration in this paper are simple, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph $G$, respectively. In this paper, we consider edge-colored graphs and let $c(uv)$ denote the color of the edge $uv$. An edge coloring of a graph is proper if the colors on edges sharing an endpoint are distinct. An edge-colored graph is rainbow if all edges have distinct colors. Rainbow matchings are of particular interest given their connection to transversals of Latin squares: each Latin square can be converted to a properly edge-colored complete bipartite graph, and a transversal of the Latin square is a rainbow perfect matching in the graph. Ryser’s conjecture [7] that every Latin square of odd order has a transversal can be seen as the beginning of the study of rainbow

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matchings. Stein [8] later conjectured that every Latin square of order \( n \) has a transversal of size \( n - 1 \); equivalently every proper edge-coloring \( K_{n,n} \) has a rainbow matching of size \( n - 1 \). The connection between Latin transversals and rainbow matchings in \( K_{n,n} \) has inspired additional interest in the study of rainbow matchings in triangle-free graphs. A thorough survey of rainbow matching and other rainbow subgraphs in edge-colored graphs appears in [5].

Several results have been attained for rainbow matchings in arbitrarily edge-colored graphs. The color degree of a vertex \( v \) in an edge-colored graph \( G \), written \( \hat{d}(v) \), is the number of distinct colors on edges incident to \( v \). We let \( \hat{\delta}(G) \) denote the minimum color degree among the vertices in \( G \). Wang and Li [10] proved that every edge-colored graph \( G \) contains a rainbow matching of size at least \( \lceil \frac{5\hat{\delta}(G) - 3}{12} \rceil \), and conjectured that a rainbow matching of size \( \lceil \hat{\delta}(G)/2 \rceil \) exists if \( \hat{\delta}(G) \geq 4 \). LeSaulnier et al. [6] then proved that every edge-colored graph \( G \) contains a rainbow matching of size \( \lceil \hat{\delta}(G)/2 \rceil \). Finally, Kostochka and Yancey [4] proved the conjecture of Wang and Li in full, and also that triangle-free graphs have rainbow matchings of size \( \lceil 2\hat{\delta}(G)/3 \rceil \).

Since the edge-colored graphs generated by Latin squares are properly edge-colored, it is of interest to consider rainbow matchings in properly edge-colored graphs. In this direction, LeSaulnier et al. proved that a properly edge-colored graph \( G \) satisfying \( |V(G)| \neq \hat{\delta}(G) + 2 \) that is not \( K_4 \) has a rainbow matching of size \( \lceil \hat{\delta}(G)/2 \rceil \). Wang then asked if there is a function \( f(\delta) \) such that a properly edge-colored graph \( G \) with minimum degree \( \delta \) and order at least \( f(\delta) \) must contain a rainbow matching of size \( \delta \) [9]. As a first step towards answering this question, Wang showed that a graph \( G \) with order at least \( 8\delta/5 \) has a rainbow matching of size \( \lceil 3\hat{\delta}(G)/5 \rceil \).

Since there are \( n \times n \) Latin squares with no transversals when \( n \) is even (see [1, 11]), for such a function \( f \) it is clear that \( f(\delta) > 2\delta \) when \( \delta \) is even. Furthermore, since maximum matchings in \( K_{\delta,n-\delta} \) have only \( \delta \) edges (provided \( n \geq 2\delta \)), there is no function for the order of \( G \) depending on \( \hat{\delta}(G) \) that can guarantee a rainbow matching of size greater than \( \hat{\delta}(G) \).

In this paper we answer Wang’s question from [9] in the affirmative.

**Theorem 1.** If \( G \) is a properly edge-colored graph satisfying \( |V(G)| \geq 13\hat{\delta}(G)/3 - 2 \), then \( G \) contains a rainbow matching of size \( \hat{\delta}(G) \).

The proof of Theorem 1 utilizes extremal techniques akin to those that appear in [4, 6, 9] and [10]. We also implement a greedy algorithmic approach to demonstrate that it is possible to efficiently construct a rainbow matching of size \( \delta \) in a properly edge-colored graph with
minimum degree $\delta$ having order at least $6.5\delta$. To our knowledge, an algorithmic approach of this type has not been previously employed in the study of rainbow matchings.

**Theorem 2.** If $G$ is a properly edge-colored graph with minimum degree $\delta$ satisfying $|V(G)| > \frac{13}{2} \delta - \frac{23}{2} + \frac{41}{82}$, then there is an $O(\delta(G)|V(G)|^2)$-time algorithm that produces a rainbow matching of size $\delta$ in $G$.

2 Proof of Theorem 1

Let $G$ be a properly edge-colored $n$-vertex graph with minimum degree $\delta$ and $n \geq \frac{13\delta}{3} - 2$. The theorem trivially holds if $\delta = 1$, so we may assume that $\delta \geq 2$. By way of contradiction, let $G$ be a counterexample with $\delta$ minimized; thus $G$ does not contain a rainbow matching of size $\delta$. Further, we may assume that $|E(G)|$ is minimized, so in particular the vertices of degree greater than $\delta$ form an independent set, as otherwise we could delete an edge without lowering the minimum degree. We break the proof into a series of simple claims.

Let $\Delta(G) = d_1 \geq d_2 \geq \ldots \geq d_n = \delta$ with $d(v_i) = d_i$ be the degree sequence of $G$.

**Lemma 3.** For $1 \leq k \leq 2\delta/3$, there exists an $i \leq k$ such that $d_i \leq 3\delta - k - 2i$.

**Proof.** Suppose that for some $k \leq 2\delta/3$, $d_i \geq 3\delta + 1 - k - 2i$ for all $1 \leq i \leq k$. Delete the vertices $v_1, v_2, \ldots, v_k$ from $G$, and note that $\delta(G \setminus \{v_1, \ldots, v_k\}) \geq \delta - k$, so there exists a rainbow matching $M_k$ on $\delta - k$ edges in $G \setminus \{v_1, \ldots, v_k\}$ by the minimality of $G$.

The vertex $v_k$ has at most $2(\delta - k)$ neighbors in $M_k$, and is incident to at most $\delta - k$ edges colored with a color occurring in $M_k$. Thus, $v_k$ has a neighbor $w_k \notin V(M_k)$ such that the color of $v_kw_k$ does not occur in $M_k$, so we can extend $M_k$ to $M_{k-1}$ by adding the edge $v_kw_k$. Note that $w_k \neq v_i$ for $i \leq k$ as $\{v_1, \ldots, v_k\}$ contains no vertices of degree $\delta$ and is therefore an independent set.

Continuing, $v_{k-1}$ has at most $2(\delta - k) + 1$ neighbors in $M_{k-1}$, and is incident to at most $\delta - k + 1$ edges colored with a color occurring in $M_{k-1}$. Thus, we can find $w_{k-1} \in N(v_{k-1})$ to extend $M_{k-1}$ to $M_{k-2}$. Continuing backwards, we can extend the matching step by step to matchings $M_i$ all the way to $M_0$, which is a rainbow matching of size $\delta$, a contradiction finishing the proof.

As a corollary of Lemma 3, we get the following lemma.

**Lemma 4.** For $1 \leq k \leq 2\delta/3$, we have $\sum_{i=1}^{k} d_i \leq k(3\delta - 2 - k)$, with equality only if $d_1 = d_k = 3\delta - 2 - k$. 

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Proof. We proceed by induction on $k$. For $k = 1$, the statement follows from Lemma 3. Now let $k > 1$ and let $i \leq k$ such that $d_i \leq 3\delta - k - 2i$. Then by induction,

$$\sum_{j=1}^{i-1} d_j \leq (i-1)(3\delta - 1 - i),$$

and

$$\sum_{j=i}^{k} d_j \leq (k-i+1)d_i \leq (k-i+1)(3\delta - k - 2i).$$

Thus,

$$\sum_{j=1}^{k} d_j \leq (i-1)(3\delta-1-i)+(k-i+1)(3\delta-k-2i) = 3k\delta - k^2 - k + 1 - i(k + 2 - i) \leq k(3\delta - 2 - k),$$

where equality can hold only for $i = 1$ and $d_k = d_1$.

Claim 5. The largest color class in $G$ contains at least two edges.

Proof. It suffices to prove that $G$ has a matching of size $\delta$, since two edges in such a matching must have the same color. Let $G_1$ be a component of $G$. Then either $G_1$ contains a path of length $2\delta - 1$, or a hamiltonian path. This is an easy consequence of the standard proof (see e.g. [2]) of Dirac’s minimum degree condition for hamiltonian cycles [3].

A path of length $2\delta - 1$ contains a matching of size $\delta$, so we may assume that every component of $G$ has hamiltonian path of length at most $2\delta - 2$. As $|G| > 2\delta - 1$, $G$ must contain at least two components. Since every component contains at least $\delta + 1$ vertices, the hamiltonian paths in each component contain matchings of size at least $\delta/2$, combining to a matching of size $\delta$.

Let $C$ be a maximum color class in $G$ and let $|C| = a$. By the minimality of $G$, there exists a rainbow matching $M = \{x_iy_i : 1 \leq i \leq \delta - 1\}$ of size $\delta - 1$ in $G - C$. Without loss of generality, we may assume that $c(x_iy_i) = i$ for $1 \leq i \leq \delta - 1$ and the edges in $C$ have color $\delta$. Let $W = V(G)\setminus V(M)$; observe that $|W| = n - 2(\delta - 1)$. If there is an edge $c$ in $G[W]$ with $c(e) \notin \{1, \ldots, \delta - 1\}$ then we can add $e$ to $M$ to obtain a rainbow matching of size $\delta$. Thus we may assume that every edge whose color is not in $\{1, \ldots, \delta - 1\}$ has an endpoint in $V(M)$. We say that an edge $uv$ is good if its color is not in $\{1, \ldots, \delta - 1\}$ and one of its endpoints is in $W$. A vertex $v \in V(M)$ is good if $v$ is incident with at least seven good edges.

Claim 6. For $i \in \{1, \ldots, \delta - 1\}$, if $x_i$ is incident with at least three good edges, then no good edge is incident with $y_i$, and vice versa.
Proof. Suppose that $y_iu$ is a good edge. If $x_i$ is incident with at least three good edges, then $x$ has a neighbor $w$ such that $vw$ is a good edge, $w \neq u$, and $c(x_iw) \neq c(y_iu)$. Thus $(M \cup \{x_iw, y_iu\})\{x_iy_i\}$ is a rainbow matching of size $\delta$, a contradiction. 

By Claim 6, we may assume without loss of generality that $\{x_1, \ldots, x_r\}$ is the set of good vertices for some $r \geq 0$. Let $W' = W \cup \{y_1, \ldots, y_r\}$.

Claim 7. No edge $uv$ in $G[W']$ has color in $\{1, \ldots, r\}$.

Proof. By way of contradiction, assume that there is an edge $uv$ in $G[W']$ such that $c(uv) \in \{1, \ldots, r\}$. Let $M'$ be the subset of $M$ consisting of the edge with color $c(uv)$ and any edges with an endpoint in $\{u, v\}$. There are at most three such edges (the edge with color $c(uv)$ and possibly one for each endpoint); without loss of generality, let $M' = \{x_1y_1, \ldots, x_iy_i\}$ (here $1 \leq t \leq 3$). Note that $x_j$ is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that $x_jw_j$ is a good edge for $1 \leq j \leq t$ and the colors on the edges $uv, x_1w_1, \ldots, x_tw_t$ are distinct. Thus $(M \cup \{uv, x_1w_1, \ldots, x_tw_t\})\{x_1y_1, \ldots, x_iy_i\}$ is a rainbow matching of size $\delta$, a contradiction.

We say that the edge $uv$ is nice if its color is not in $\{r + 1, \ldots, \delta - 1\}$ and one of its endpoints is in $W'$. Note that every good edge is nice. Recall that every good edge has an endpoint in $V(M)$. By Claim 6 and Claim 7, no nice edge lies in $G[W']$. Hence, every nice edge joins vertices in $W'$ and $V(G) \setminus W'$. A vertex $v \in V(M) \setminus \{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ is nice if $v$ is incident with at least seven nice edges. Note that if there is no good vertex (i.e. $r = 0$), then the definitions of good and nice vertices are the same and so there is also no nice vertex. Next, we show the analogue of Claim 6 and Claim 7 for nice vertices and edges.

Claim 8. For $i \in \{r + 1, \ldots, \delta - 1\}$, if $x_i$ is incident with at least three nice edges, then no nice edge is incident with $y_i$, and vice versa.

Proof. Suppose $y_iu$ is a nice edge for some $i \in \{r + 1, \ldots, \delta - 1\}$. If $x_i$ is incident to at least three nice edges, then $x_i$ has a neighbor $v$ such that $x_iv$ is a nice edge, $v \neq u$, and $c(x_iv) \neq c(y_iu)$. Let $M'$ be the subset of $M$ consisting of edges with an endpoint in $\{u, v\}$ or a color in $\{c(x_iv), c(y_iu)\}$. There are at most four such edges (possibly one with each endpoint and one with each color); without loss of generality, let $M' = \{x_1y_1, \ldots, x_iy_i\}$ (here $0 \leq t \leq 4$). Note that $x_j$ is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that $x_jw_j$ is a good edge for $1 \leq j \leq t$ and the colors on the edges $x_iv, y_iu, x_1w_1, \ldots, x_tw_t$ are distinct. Thus $(M \cup \{x_iy_iu, x_1w_1, \ldots, x_tw_t\})\{x_iy_i, x_1y_1, \ldots, x_iy_t\}$ is a rainbow matching of size $\delta$, a contradiction. 

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By Claim 8, we may assume that \( \{x_{r+1}, x_{r+2}, \ldots, x_{r+s}\} \) is the set of nice vertices for some \( s \geq 0 \).

**Claim 9.** No edge \( uv \) in \( G[W'] \) has color in \( \{1, \ldots, r+s\} \).

*Proof.* By Claim 7, the claim holds if \( s = 0 \). Assume that \( s \geq 1 \), and consequently \( r \geq 1 \). Without loss of generality, suppose that there is an edge \( uv \) in \( G[W'] \) with \( c(uv) = r + 1 \). Because \( x_{r+1} \) is nice, it has a neighbor \( v' \) in \( W' \) such that \( x_{r+1}v' \) is a nice edge and \( v' \neq u, v \).

The rest of the proof is essentially the same as the proofs of Claims 7 and 8. Let \( M' \) be the subset of \( M \) consisting of those edges an endpoint in \( \{u, v, v'\} \) or color \( c(x_{r+1}v') \). Again there are at most four edges in \( M' \) and we let \( M' = \{x_1y_1, \ldots, x_ty_t\} \). Defining \( w_1, \ldots, w_t \) as before, \((M \cup \{vw, x_{r+1}v', x_1w_1, \ldots, x_tw_t\}) \setminus \{x_{r+1}y_{r+1}, x_1y_1, \ldots, x_ty_t\} \) is a rainbow matching of size \( \delta \), a contradiction. \( \square \)

Next, we count the number of nice edges in \( G \).

**Claim 10.** There are at most \( \max \{(3\delta - 8 - r + s)r + 6(\delta - 1), (7\delta / 3 - 7 + s)r + 6(\delta - 1)\} \) nice edges in \( G \).

*Proof.* Recall that \( V(G) \setminus W' = \{x_1, \ldots, x_{\delta-1}, y_{r+1}, \ldots, y_{\delta-1}\} \) and every nice edge joins vertices from \( W' \) and \( V(G) \setminus W' \). For \( r \leq 2\delta/3 \), the set of good vertices is incident to at most \( r(3\delta - 2 - r) \) nice edges by Lemma 4. Similarly, for \( r > 2\delta/3 \), the set of good vertices is incident to at most \( r(3\delta - 2 - [2\delta/3]) \leq r(7\delta/3 - 1) \) nice edges. For \( i \in \{r + 1, \ldots, r + s\} \), since \( x_i \) is nice, by Claim 8 \( x_i \) is incident to at most \( r + 6 \) nice edges and \( y_i \) is incident to none. For \( i \in \{r + s + 1, \ldots, \delta - 1\} \), by Claim 8 there are at most six nice edges with an endpoint in \( \{x_i, y_i\} \). Therefore, the number of nice edges is at most

\[
(3\delta - 2 - r)r + (r + 6)s + 6(\delta - 1 - r - s) = (3\delta - 8 - r + s)r + 6(\delta - 1) \quad \text{for} \quad r \leq 2\delta/3,
\]

and

\[
(7\delta/3 - 1) r + (r + 6)s + 6(\delta - 1 - r - s) = (7\delta / 3 - 7 + s)r + 6(\delta - 1) \quad \text{for} \quad r > 2\delta/3. \quad \square
\]

Recall that \( C \) is the color class with color \( \delta \), \( |C| = a \), and \( C \) is a maximum size color class. Therefore there are at least \( 2(a - \delta + 1) \) vertices in \( W \) incident to an edge in \( C \). Since every edge in \( C \) has a matching in \( V(M) \) it follows that there are at least \( 2(a - \delta + 1) \) vertices in \( V(M) \) joined to \( W \) by edges in \( C \). Without loss of generality, let \( \{r + s + 1, \ldots, r + s + t\} \) be the set of indices \( i \in \{r + s + 1, \ldots, \delta - 1\} \) such that \( x_i \) or \( y_i \) is incident to an edge with color \( \delta \). By Claim 6 and Claim 8, we have

\[
t \geq a - \delta + 1 - \frac{r + s}{2} \quad \text{and} \quad r + s + t \leq \delta - 1. \tag{1}
\]
Claim 11. For \( i \in \{r + s + 1, \ldots, r + s + t\} \), there is at most one edge of color \( i \) in \( G[W] \).

Proof of claim. Suppose \( uv \) and \( u'v' \) are edges of color \( i \) in \( G[W] \) for some \( i \in \{r + s + 1, \ldots, r + s + t\} \). Without loss of generality, we may assume that there exists \( w \in W \) such that \( c(x_iw) = \delta \) and \( w \neq u, v \). Hence, \( (M \cup \{uv, x_iw\}) \setminus \{x_iy_i\} \) is a rainbow matching of size \( \delta \), a contradiction. \( \square \)

Now, we count the number of nice edges from \( W' \) to \( V(G) \setminus W' \). Recall that each color class in \( G \) contains at most \( a \) edges. By Claim 9, there is no edge in \( G[W'] \) of color \( i \in \{r + 1, \ldots, r + s\} \). Thus, for \( i \in \{r + 1, \ldots, r + s\} \) there are at most \( a - 1 \) vertices in \( W' \) that are incident with an edge of color \( i \). By Claim 11, for \( i \in \{r + s + 1, \ldots, r + s + t\} \), there are at most \( a \) vertices in \( W' \) that are incident with an edge of color \( i \). Recall that \( W' \setminus W = \{y_1, \ldots, y_r\} \). Hence, for \( i \in \{r + s + 1, \ldots, r + s + t\} \), there are at most \( a + r \) vertices in \( W' \) that are incident with an edge of color \( i \). Since every color class has size at most \( a \), for \( i \in \{r + s + t + 1, \ldots, \delta - 1\} \), there are at most \( 2(a - 1) \) vertices in \( W' \) that are incident with an edge of color \( i \). It then follows, using the fact that \( |W'| = |W| + r = n - 2(\delta - 1) + r \) and (1), that the number of nice edges from \( W' \) to \( V(G) \setminus W' \) is at least

\[
\begin{align*}
\delta|W'| &- (a - 1)s - (a + r)t - 2(a - 1)(\delta - 1 - r - s - t) \\
&= \delta n - 2\delta(\delta - 1) - (a - 1)(2\delta - 2 - 2r - s) + (a - 2)t + (\delta - t)r \\
&\geq \delta n - 2\delta(\delta - 1) - (a - 1)(2\delta - 2 - 2r - s) + (a - 2)t + (r + s + 1)r.
\end{align*}
\]

Now assume that \( r \leq 2\delta/3 \). Since there are at most \( (3\delta - 8 - r + s)r + 6(\delta - 1) \) nice edges in \( G \) by Claim 10,

\[
\delta n \leq (3\delta - 9 - 2r)r - (a - 2)t + 2(\delta + 3)(\delta - 1) + (a - 1)(2\delta - 2 - 2r - s). \quad (2)
\]

To finish the proof we bound the right hand side of (2) to obtain a contradiction. Note that \(-(a - 2)\), the coefficient of \( t \), is nonpositive by Claim 5. Thus the right hand side of (2) is maximized when \( t \) is minimized. By (1), \( t \geq \max\{a - \delta + 1 - (r + s)/2, 0\} \).

If \( a \leq \delta - 1 + (r + s)/2 \), then we let \( t = 0 \). The coefficient of \( a \) becomes \( 2\delta - 2 - 2r - s \geq 2(\delta - 1 - r - s) \geq 0 \). Thus (2) is maximized when \( a \) is maximized, and evaluating at \( a = \delta - 1 + (r + s)/2 \) yields

\[
\delta n \leq 2(2\delta + 1)(\delta - 1) + (2\delta - 6 - 3r)r - (3r + s - 2)s/2.
\]

Recall that if \( s \geq 1 \), then \( r \geq 1 \). Hence, \((3r + s - 2)s \geq 0 \) and so

\[
\delta n \leq 2(2\delta + 1)(\delta - 1) + (2\delta - 6 - 3r)r.
\]
This is maximized when \( r = \delta/3 - 1 \), yielding \( n \leq 13\delta/3 - 4 + 1/\delta \). Since \( \delta \geq 2 \) this is a contradiction.

If \( a \geq \delta - 1 + (r + s)/2 \), we let \( t = a - \delta + 1 - (r + s)/2 \). Then, (2) becomes

\[
\delta n \leq (3\delta - 1 - (3r + s)/2 - a)a + (3\delta - 8 - 2r)r + 2(\delta + 1)(\delta - 1).
\]

If \( (3\delta - 1)/2 - (3r + s)/4 \leq \delta - 1 + (r + s)/2 \), then the right hand side is maximized when \( a = \delta - 1 + (r + s)/2 \), which corresponds to the case when \( a \leq \delta - 1 + (r + s)/2 \) and so we are done. Letting \( a = (3\delta - 1)/2 - (3r + s)/4 \) yields

\[
\begin{align*}
\delta n & \leq \frac{1}{4} \left( 3\delta - 1 - \frac{3r + s}{2} \right)^2 + (3\delta - 8 - 2r)r + 2(\delta + 1)(\delta - 1) \\
& = \left( -\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4} \right)r + \left( \frac{3r}{8} + \frac{s}{16} - \frac{3\delta}{4} + \frac{1}{4} \right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4} \\
& = \left( -\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4} \right)r + \left( \frac{6r + s + 4 - 12\delta}{16} \right)s + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4}.
\end{align*}
\]

Since \( \delta \geq r + s + 1 \), this is maximized when \( s = 0 \), yielding

\[
\delta n \leq \left( -\frac{23r}{16} + \frac{3\delta}{4} - \frac{29}{4} \right)r + \frac{17\delta^2}{4} - \frac{3\delta}{2} - \frac{7}{4} \tag{3}
\]

The right hand side of (3) is maximized at \( r = 2(29 - 3\delta)/23 \leq 2 \), yielding

\[
n \leq 17\delta/4 - \min\{2, 22/\delta\} < 13\delta/3 - 2,
\]
a contradiction.

To complete the proof of the theorem, we are left with the case \( r > 2\delta/3 \). Similarly to (2), since we have at most \((7\delta/3 - 7 + s)r + 6(\delta - 1)\) nice edges in \( G \) by Claim 10, we have

\[
\delta n \leq (7\delta/3 - 8 - r)r - (a - 2)t + 2(\delta + 3)(\delta - 1) + (a - 1)(2\delta - 2 - 2r - s). \tag{4}
\]

Again the right hand side of (4) is maximized when \( t \) is minimized, and \( t \geq \max\{a - \delta + 1 - (r + s)/2, 0\} \).

If \( a \leq \delta - 1 + (r + s)/2 \), (4) is maximized when \( t = 0 \) and \( a = \delta - 1 + (r + s)/2 \), which yields with \((3r + s - 2)s \geq 0 \)

\[
\delta n \leq 2(2\delta + 1)(\delta - 1) + (4\delta/3 - 5 - 2r)r.
\]

This is maximized when \( r = 2\delta/3 \), yielding \( n < 4\delta \), a contradiction.
If $a \geq \delta - 1 + (r + s)/2$, we let $t = a - \delta + 1 - (r + s)/2$. Then, (4) becomes
\[
\delta n \leq (3\delta - 1 - (3r + s)/2 - a)a + (7\delta/3 - 7 - r)r + 2(\delta + 1)(\delta - 1).
\]
If $(3\delta - 1)/2 - (3r + s)/4 \leq \delta - 1 + (r + s)/2$, then the right hand side is maximized when $a = \delta - 1 + (r + s)/2$, which corresponds to the case when $a \leq \delta - 1 + (r + s)/2$ and so we are done. On the other hand, if $(3\delta - 1)/2 - (3r + s)/4 > \delta - 1 + (r + s)/2$, then
\[
10\delta/3 \leq 5r \leq 5r + 3s < 2(\delta + 1),
\]
a contradiction finishing the proof of the theorem.

3 Proof of Theorem 2

We proceed by induction on $\delta(G)$. The result is trivial if $\delta(G) = 1$. We assume that $G$ is a graph with minimum degree $\delta > 1$ and order greater than $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$.

Lemma 12. If $G$ has a color class containing at least $2\delta - 1$ edges, then $G$ has a rainbow matching of size $\delta$.

Proof. Let $C$ be a color class with at least $2\delta - 1$ edges. By induction, there is a rainbow matching $M$ of size $\delta - 1$ in $G - C$. There are $2\delta - 2$ vertices covered by the edges in $M$, thus one of the edges in $C$ has no endpoint covered by $M$, and the matching can be extended. \qed

It is also useful to note that we also have the following, which is identical to Lemma 3 when $k = 1$.

Lemma 13. If $G$ satisfies $\Delta(G) > 3\delta - 3$, then $G$ has a rainbow matching of size $\delta$.

We begin by preprocessing the graph so that each edge is incident to at least one vertex with degree $\delta$. To achieve this, arbitrarily order the edges in $G$ and process them in order. If both endpoints of an edge have degree greater than $\delta$ when it is processed, delete that edge. In the resulting graph, every edge is incident to a vertex with degree $\delta$. Furthermore, by Lemma 13 we may assume that $\Delta(G) \leq 3\delta - 3$; thus the degree sum of the endpoints of any edge is bounded above by $4\delta - 3$. After preprocessing, we begin the greedy algorithm.

In the $i$th step of the algorithm, a smallest color class is chosen (without loss of generality, color $i$), and then an edge $e_i$ of color $i$ is chosen such that the degree sum of the endpoints is minimized. All the remaining edges of color $i$ and all edges incident with the endpoints of $e_i$ are deleted. The algorithm terminates when there are no edges in the graph.
We assume that the algorithm fails to produce a matching of size $\delta$ in $G$; suppose that the rainbow matching $M$ generated by the algorithm has size $k$. We let $R$ denote the set of vertices that are not covered by $M$.

Let $c_i$ denote the size of the smallest color class at step $i$. Since at most two edges of color $i + 1$ are deleted in step $i$ (one at each endpoint of $e_i$), we observe that $c_{i+1} + 2 \geq c_i$. Otherwise, at step $i$ color class $i + 1$ has fewer edges. Let step $h$ be the last step in the algorithm in which a color class that does not appear in $M$ is completely removed from $G$. It then follows that $c_h \leq 2$, and in general $c_i \leq 2(h - i + 1)$ for $i \in [h]$. Let $f_i$ denote the number of edges of color $i$ deleted in step $i$ with both endpoints in $R$. Since $f_i < c_i$, we have $f_i \leq 2(h - i) + 1$ for $i \in [h]$. Note that after step $h$, there are exactly $k - h$ colors remaining in $G$. By Lemma 12, color classes contain at most $2\delta - 2$ edges, and therefore the last $k - h$ steps remove at most $(k - h)(2\delta - 2)$ edges. Furthermore, for $i > h$, the degree sum of the endpoints of $e_i$ is at most $2(\delta - 1)$.

For $i \in [h]$, let $x_i$ and $y_i$ be the endpoints of $e_i$, and let $d_i(v)$ denote the degree of a vertex $v$ at the beginning of step $i$. Let $\mu_i = \max\{0, d_i(x_i) + d_i(y_i) - 2\delta\}$; note that $2\delta \leq 2\delta + \mu_i \leq 4\delta - 3$. Thus, at step $i$, at most $2\delta + \mu_i + f_i - 1$ edges are removed from the graph. Since the algorithm removes every edge from the graph, we conclude that

$$|E(G)| \leq (k - h)(2\delta - 2) + \sum_{i=1}^{h} (2\delta + \mu_i + f_i - 1).$$  \hspace{1cm} (5)

We now compute a lower bound for the number of edges in $G$. Since the degree sum of the endpoints of $e_i$ in $G$ is at least $2\delta + \mu_i$, we immediately obtain the following inequality:

$$\frac{n\delta + \sum_{i \in [k]} \mu_i}{2} \leq |E(G)|.$$

If $f_i > 0$ and $\mu_i > 0$, then there is an edge with color $i$ having both endpoints in $R$. Since this edge was not chosen in step $i$ by the algorithm, the degree sum of its endpoints is at least $2\delta + \mu_i$, and one of its endpoints has degree at least $\delta + \mu_i$. For each value of $i$ satisfying $f_i > 0$, we wish to choose a representative vertex in $R$ with degree at least $\delta + \mu_i$. Since there are $f_i$ edges with color $i$ having both endpoints in $R$, there are $f_i$ possible representatives for color $i$. Since a vertex in $R$ with high degree may be the representative for multiple colors, we wish to select the largest system of distinct representatives.

Suppose that the largest system of distinct representatives has size $t$, and let $T$ be the set of indices of the colors that have representatives. For each color $i \in T$ there is a distinct vertex in $R$ with degree at least $\delta + \mu_i$. Thus we may increase the edge count of $G$ as follows:
We let $\{f_i^+\}$ denote the sequence $\{f_i\}_{i \in [h]}$ sorted in nonincreasing order. Since $f_i \leq 2(h - i) + 1$, we conclude that $f_i^+ \leq 2(h - i) + 1$. Because there is no system of distinct representatives of size $t + 1$, the sequence $\{f_i^+\}$ cannot majorize the sequence $\{t + 1, t, t - 1, \ldots, 1\}$. Hence there is a smallest value $p \in [t + 1]$ such that $f_p^+ \leq t + 1 - p$. Therefore, the maximum of $\sum_{i=1}^h f_i^+$ is bounded by the sum of the sequence $\{2h - 1, 2h - 3, \ldots, 2(h - p) + 3, t + 1 - p, \ldots, t + 1 - p\}$. Summing we attain

$$\sum_{i \in [h]} f_i \leq (p - 1)(2h - p + 1) + (h - p + 1)(t + 1 - p).$$

Over $p$, this value is maximized when $p = t + 1$, yielding $\sum_{i \in [h]} f_i \leq t(2h - t)$. Since $h \leq \delta - 1$, we then have $\sum_{i \in [h]} f_i \leq 2(\delta - 1)t - t^2$.

We now combine bounds (5) and (6):

$$\frac{n\delta + \sum_{i \in [h]} \mu_i + \sum_{i \in T} \mu_i}{2} \leq (k - h)(2\delta - 2) + \sum_{i=1}^h (2\delta + \mu_i + f_i - 1).$$

Hence, since $k \leq \delta - 1$,

$$\frac{n\delta}{2} \leq (2\delta - 1)(\delta - 1) + \frac{1}{2} \sum_{i \in [h]} \mu_i + \sum_{i \in [h]} f_i \leq (2\delta - 1)(\delta - 1) + (\delta - 1 - t)(\delta - 3/2) + 2(\delta - 1)t - t^2 \leq 3\delta^2 - \frac{11}{2}\delta + \frac{5}{2} + \left(\delta - \frac{1}{2}\right)t - t^2.$$

This bound is maximized when $t = (\delta - \frac{1}{2})/2$. Thus

$$n \leq \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

contradicting our choice for the order of $G$.

It remains to show that the proof given above provides the framework of a $O(\delta(G)|V(G)|^2)$-time algorithm that generates a rainbow matching of size $\delta(G)$ in a properly edge-colored graph $G$ of order at least $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$. Given such a $G$, we create a sequence of graphs $\{G_i\}$ as follows, letting $G = G_0$, $\delta = \delta(G)$, and $n = |V(G)|$. First, determine $\delta(G_i)$, $\Delta(G_i)$, and the maximum size of a color class in $G_i$; this process takes $O(n^2)$-time. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ and the maximum color class has at most $2\delta(G_i) - 2$ edges, then terminate the sequence and

$$n\delta + \sum_{i \in [h]} \mu_i + \sum_{i \in T} \mu_i \leq |E(G)|. \quad (6)$$
set $G_i = G'$. If $\Delta(G_i) > 3\delta(G_i) - 3$, then delete a vertex $v$ of maximum degree and then process the edges of $G_i - v$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is $G_{i+1}$. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ but a maximum color class $C$ has at least $2\delta(G_i) - 1$ edges, then delete $C$ and then process the edges of $G_i - C$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is $G_{i+1}$. Note that $\delta(G_{i+1}) = \delta(G_i) - 1$. If this process generates $G_\delta$, we set $G' = G_\delta$ and terminate. Generating the sequence $\{G_i\}$ consists of at most $\delta$ steps, each taking $O(n^2)$-time.

Given that $G' = G_i$, the algorithm from the proof of Theorem 2 takes $O(\delta n^2)$-time to generate a matching of size $\delta - i$ in $G'$. The preprocessing step and the process of determining a smallest color class and choosing an edge in that class whose endpoints have minimum degree sum both take $O(n^2)$-time. This process is repeated at most $\delta$ times.

A matching of size $\delta - (i + 1)$ in $G_{i+1}$ is easily extended in $G_i$ to a matching of size $\delta - i$ using the vertex of maximum degree or maximum color class. The process of extending the matching takes $O(\delta)$-time. Thus the total run-time of the algorithm generating the rainbow matching of size $\delta$ in $G$ is $O(\delta n^2)$.

It is worth noting that the analysis of the greedy algorithm used in the proof of Theorem 2 could be improved. In particular, the bound $c_{i+1} \geq c_i - 2$ is sharp only if at step $i$ there are an equal number of edges of color $i$ and $i + 1$ and both endpoints of $e_i$ are incident to edges with color $i + 1$. However, since one of the endpoints of $e_i$ has degree at most $\delta$, at most $\delta - 1$ color classes can lose two edges in step $i$. Since the maximum size of a color class in $G$ is at most $2\delta - 2$, if $G$ has order at least $6\delta$, then there are at least $3\delta/2$ color classes. Thus, for small values of $i$, the bound $c_i \leq 2(k - i + 1)$ can likely be improved. However, we doubt that such analysis of this algorithm can be improved to yield a bound on $|V(G)|$ better than $6\delta$.

References


