Extremal Graphs Having No Stable Cutsets

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Abstract

A stable cutset in a graph is a stable set whose deletion disconnects the graph. It was conjectured by Caro and proved by Chen and Yu that any graph with \( n \) vertices and at most \( 2n - 4 \) edges contains a stable cutset. The bound is tight, as we will show that all graphs with \( n \) vertices and \( 2n - 3 \) edges without stable cutset arise recursively glueing together triangles and triangular prisms along an edge or triangle. As a by-product, an algorithmic implication of our result will be pointed out.

Keywords: stable cutset; independent cutset; fragile graph; extremal graph

1 Introduction

All graphs considered are finite and have no loops or multiple edges. For a graph \( G = (V(G), E(G)) \) with vertex set \( V(G) \) and edge set \( E(G) \), write \( |G| = |V(G)| \) and \( \|G\| = |E(G)| \). A stable set (or an independent set) in \( G \) is a set of pairwise non-adjacent vertices. A cutset (or separator) of \( G \) is a set \( S \) of vertices such that \( G - S \) is disconnected. A stable cutset in \( G \) is a cutset of \( G \) which is also a stable set. Throughout this paper, we consider the empty set as stable. So in particular, every disconnected graph contains a stable cut set. It is naturally expected that graphs with few edges would have stable cutsets. Indeed, the following theorem was conjectured by Caro and proved by Chen and Yu.

**Theorem 1** ([4]). Let \( G \) be a graph with \( \|G\| \leq 2|G| - 4 \). Then \( G \) contains a stable cutset.

Small stable cutsets are discussed in [3], and algorithmic and complexity aspects of stable cutsets are discussed in [5, 1, 7, 8, 9]. The importance of stable cutsets in connection
to perfect graphs are demonstrated in [6, 11]. In [2] it is noted that graphs containing
stable cutsets play a role in some decomposition algorithms.
Actually, Chen and Yu proved the following stronger result.

**Theorem 2 ([4]).** Let $G$ be 2-connected a graph with $\|G\| \leq 2|G| - 4$. Then for every
vertex $x \in V(G)$, there is a stable cutset not containing $x$.

This implies immediately the following corollary; a vertex $x$ is a cut vertex if $\{x\}$ is a
cutset.

**Corollary 3.** Let $G$ be a graph with $\|G\| \leq 2|G| - 4$, and $x \in V(G)$. Unless $x$ is the
unique cut vertex in $G$, there is a stable cutset not containing $x$.

The bound in Theorem 1 is tight. In the next section we describe all graphs with
$n$ vertices and $2n - 3$ edges that have no stable cutset (Theorem 5). In the last section we
will point out an algorithmic implication of our result.

**Notation and definitions.** Let $G$ be a graph. The complement of $G$ is written $\overline{G}$.
The neighborhood of a vertex $v$ in $G$, denoted by $N_G(v)$, is the set of all vertices in $G$
adjacent to $v$; if the context is clear, we simply write $N(v)$. Set $\deg(v) = |N(v)|$, the
degree of the vertex $v$. For a subset $W \subseteq V(G)$, $N(W) = \bigcup_{w \in W} N(w) \setminus W$, and $G[W]$ is
the subgraph of $G$ induced by $W$; write $G - W = G[V(G) \setminus W]$ and $G - w = G - \{w\}$.
Given another graph $H$, an $H$-cutset $S$ of $G$ is a cutset such that $G[S]$ is isomorphic to
$H$, while a $k$-cutset is a $k$-element cutset. An edge cut of $G$ is a set $M$ of edges such that
$G - M = (V(G), E(G) \setminus M)$ is disconnected. A matching cut in $G$ is an edge cut of $G$
that is also a matching.

$P_k$ stands for the path with $k$ vertices and $k - 1$ edges, $C_k$ is the cycle with $k$ vertices
and $k$ edges. A complete graph with $k$ vertices is denoted $K_k$; $K^-k$ is $K_k$ minus an edge.
The $K_3$ is also called a triangle and the $\overline{C_6}$ is also called a triangular prism; see Figure 1.

\[ 
\begin{array}{c}
\text{Figure 1: The triangular prism $\overline{C_6}$}.
\end{array}
\]

We will make use of the following well-known graph operation. A clique in a graph is a
set of pairwise adjacent vertices. Let $G_1, G_2$ be disjoint graphs which each have nonempty
cliques $Q_1$, respectively, $Q_2$ of the same size. A graph obtained from $G_1$ and $G_2$ by first
choosing a bijection $f : Q_1 \rightarrow Q_2$ and then identifying each $x$ in $Q_1$ with $f(x)$ in $Q_2$
is said to arise from $G_1$ and $G_2$ by clique identification. If the chosen cliques have two,
respectively, three vertices, we also speak of edge identifications, respectively, triangle
identifications. Finally, for convenience, we consider $G_1$ and $G_2$ as induced subgraphs of
the graph arising from $G_1$ and $G_2$ by clique identification. Thus, a graph $G$ arises from
two graphs by clique identification if and only if there exist induced subgraphs $G_1$ and
$G_2$ in $G$ such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is a clique.
2 The Result

Let $G_{sc}$ be the class of graphs one gets by recursively glueing together triangles and triangular prisms along an edge or triangle. More precisely,

1. $K_3 \in G_{sc}$ and $C_6 \in G_{sc}$.
2. If $G_1, G_2 \in G_{sc}$ and $G$ is obtained from $G_1$ and $G_2$ by edge identification, then $G \in G_{sc}$.
3. If $G_1, G_2 \in G_{sc}$ and $G$ is obtained from $G_1$ and $G_2$ by triangle identification, then $G \in G_{sc}$.

Notice that we may restrict to $G_2 \in \{K_3, C_6\}$ in the above definition without changing the class $G_{sc}$, which effects the complexity of the algorithm considered in the last section of the paper.

Proposition 4. Any graph $G \in G_{sc}$ has $\|G\| = 2|G| - 3$ edges and no stable cutset.

Proof. The statement is obvious for triangles and triangular prisms. Let $G$ arise from $G_1, G_2 \in G_{sc}$ by edge or triangle identification, and write $G = G_1 \cup G_2$ with clique $Q = G_1 \cap G_2$ of size two or three. Then

$$|G| = |G_1| + |G_2| - |Q| \quad \text{and} \quad \|G\| = \|G_1\| + \|G_2\| - \|Q\|,$$

and hence, by induction,

$$\|G\| = (2|G_1| - 3) + (2|G_2| - 3) - \|Q\| = (2|G| - 3) + (2|Q| - 3 - \|Q\|) = 2|G| - 3.$$

Note that, as $Q$ is a clique, any stable cutset in $G$ is also a stable cutset in $G_1$ or $G_2$. Hence, by induction again, $G$ has no stable cutset.

Theorem 5. Let $G$ be a graph with $\|G\| \leq 2|G| - 3$. Then $G$ contains a stable cutset or $G \in G_{sc}$.

Proof. Our proof starts with a number of claims. For the sake of contradiction, we assume that $G$ is a minimal counterexample to Theorem 5. Then, by Theorem 1,

Claim 6. $\|G\| = 2|G| - 3$.

Claim 7. Every vertex $v$ lies in a triangle.

Otherwise, $N(v)$ would be a stable cutset in $G$.

Claim 8. $G$ contains no $K_2$-cutset and no $K_3$-cutset.

Otherwise, let $G$ contain a cutset $Q$ isomorphic to $K_2$ or $K_3$. Write $G = G_1 \cup G_2$ with $G_1 \cap G_2 = Q$. Since $G$ has no stable cutset and $Q$ is a clique, $G_1$ and $G_2$ have no stable cutset. By Theorem 1, $\|G_i\| \geq 2|G_i| - 3$, $i = 1, 2$, hence, by Claim 6, $\|G_i\| = 2|G_i| - 3$. Therefore, by the minimality of $G$, $G_i \in G_{sc}$, and thus $G \in G_{sc}$, a contradiction.
Claim 9. $G$ is 3-connected.

Otherwise, by Claim 8, $G$ would contain a stable cutset.

Claim 10. $G$ contains no 3-edge matching cut.

Otherwise, let $M = \{x_1y_1, x_2y_2, x_3y_3\}$ be a matching cut of $G$. Since $G$ is 2-connected, $G - M$ has exactly two components, say $G_1$ and $G_2$. Then the set of all edges between $G_1$ and $G_2$ is exactly $M$, and we may assume that $x_1, x_2, x_3 \in V(G_1)$, $y_1, y_2, y_3 \in V(G_2)$. Now, if $\{x_1, x_2, x_3\}$ is not a clique, say $x_1x_2 \notin E(G)$, then $\{x_1, x_2, y_3\}$ is a stable cutset of $G$, a contradiction. So, $\{x_1, x_2, x_3\}$ and, by symmetry, $\{y_1, y_2, y_3\}$ are cliques. Since $G \neq C_6$, at least one of these cliques must be a cutset of $G$, contradicting Claim 8.


Otherwise, contract the edge between the two vertices of degree 3 in this (not necessarily induced) subgraph, resulting in a graph $G'$. By Claim 9, $G'$ is 2-connected. By Claim 6, $\|G'\| \leq \|G\| - 3 = 2|G| - 6 = 2|G'|-4$. By Theorem 2, $G'$ contains a stable cutset not containing the new vertex, which is also a stable cutset in $G$.

Claim 12. For any two non-adjacent vertices $x, y$ we have $|N(x) \cap N(y)| \leq 2$.

Otherwise, contract the two vertices, and get a (2-connected) graph $G'$ with $\|G'\| \leq 2|G'|-4$. Then, $G'$ has a stable cutset by Theorem 1 which yields a stable cutset in $G$.


Otherwise, let $\{x, y, z\}$ be a cutset of $G$ such that $xy, yz \in E(G)$, and let $G_1$ and $G_2$ be induced subgraphs of $G$ with $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{x, y, z\}$. Then

$$\|G_1\| + \|G_2\| = \|G\| + 2 = 2|G| - 1 = 2|G_1| + 2|G_2| - 7.$$ 

Thus, by symmetry $\|G_1\| \leq 2|G_1| - 4$, and note that, by Claim 9, $y$ is not a cut vertex of $G_1$. Therefore, by Corollary 3, $G_1$ contains a stable cutset not containing $y$. But this is then also a stable cutset in $G$.

Claim 14. In every triangle, at least two vertices belong to other triangles as well.

Proof of Claim 14: Assume that $xyz$ is a triangle and $y$ and $z$ are in no other triangles. Then there is an edge $y'z'$ with $y' \in N(y)$ and $z' \in N(z)$ as otherwise, by Claim 11, $(N(y) \cup N(z)) \setminus \{y, z\}$ is a stable cutset. Contracting $\{y, z\}$ to a new vertex $v$ yields a graph $G'$ with $\|G'\| \leq 2|G'|-3$. Since every stable cutset in $G'$ yields a stable cutset in $G$, $G'$ has no stable cutset. So $G' \in G_{sc}$ by the minimality of $G$.

Now assume that $G'$ contains a 3-edge matching cut $M$. Then by Claim 10, $v$ is in one of the edges in $M$, say $M = \{au, bv, cw\}$, and further, we have $by', bz \in E(G)$. Let the two triangles in $G'$ enclosing $M$ be $abc$ and $uvw$, where $b \in \{y', z\}$ by Claim 12.

Let $G_1$ be the component of $G - \{au, by', bz, cw\}$ containing $abc$. By Claim 8, $G_1 = abc$. Let $G_2 = G - G_1$, and note that $\|G_2\| = 2|G_2| - 4$. Further, $G_2$ is 2-connected as any 1-cutset in $G_2$ would yield a 2-cutset in $G$ if we add $b$ to it. We may assume by symmetry that $z'w \in E(G)$ (we will not use the fact that $y$ lies in only one triangle, so $y$ and $z'$ are
symmetric in the following argument). Further assume that $yu \in E(G)$. By Theorem 2, $G_2$ contains a stable cutset $X$ not containing $w$. As $X$ is not a stable cutset of $G$, $y$ is in a different component of $G_2 - X$ than $w$. Thus, $u \in X$ and $X \cup \{b\}$ is a cutset of $G$, and as this is not a stable cutset, $z' \in X$. But then $X \cup \{c\}$ is a stable cutset in $G$, a contradiction. So $yu \notin E(G)$ and therefore $z'u \in E(G)$. By Theorem 2, $G_2$ contains a stable cutset $X$ not containing $z'$, with $y$ and $z'$ in different components of $G_2 - X$. But then $X \cup \{b\}$ is a stable cutset in $G$, a contradiction. Therefore, $G'$ contains no 3-edge matching cut.

As a result, $G'$ can be built by starting with a triangle and recursively glueing on triangles along an edge ($G'$ is a so-called 2-tree). As $G$ is 3-connected, every such 2-cutset in $G'$ has the form $uv$, and $u$ is connected to exactly one of $y$ and $z'$ by Claim 13. Further, every vertex of degree at least 3 in $G'$ lies in such a cutset.

On the other hand, there are at least two vertices of degree 2 in $G'$. As $G$ is 3-connected, such vertices must lie in $N_G(y) \cap N_G(z')$, but by Claim 12, $N_G(y) \cap N_G(z') = \{y', z\}$. Thus, exactly the two vertices $y'$ and $z$ have degree 2 in $G'$, and $N_G(z) = \{x, y, z'\}$. By a symmetric argument using a contraction of $\{y', z\}$ instead of $\{y, z'\}$ in the beginning, $N_G(y) = \{x, y', z\}$. But this implies that $N_G(z') = V(G) \setminus \{y, z'\}$, as every vertex in $V(G) \setminus \{y, z'\}$ is in $N_G(y) = N_G(y) \cup N_G(z')$. In particular, $x z' \in E(G)$. This contradicts Claim 11, as $G[\{x, y, z, z'\}]$ is then a $K_4^-$, hence Claim 14 follows.

Consider the vertex-triangle incidence graph $H$ of $G$, i.e., the bipartite graph with partite sets $V(G)$ and the set of all triangles $T(G)$ in $G$, with an edge between a vertex $v \in V(G)$ and a triangle $T \in T(G)$ if $v \in V(T)$. By Claim 14, $H$ is not a tree.

Let $x_1T_1x_2T_2\ldots x_kT_kx_1$ be a shortest cycle in $H$. By Claim 11, $H$ has no cycles of length less or equal to 6, so $k \geq 4$. Then $C = x_1x_2\ldots x_kx_1$ is a cycle in $G$, and $V(T_i) \setminus V(C)$ consists of a distinct vertex for every $1 \leq i \leq k$.

If we contract $P = x_1x_2\ldots x_{k-1}$ to a new vertex $v$, we get a graph $G'$ with $\|G'\| \leq 2|G'|-4$. If $v$ is not the unique cut vertex of $G'$, then we can use Corollary 3 to find a stable cutset of $G'$ not containing $v$, which is then also a stable cutset of $G$, a contradiction. Thus, $v$ is the unique cut vertex of $G'$. Let $Y$ be a component of $G \setminus V(P)$ and $1 \leq r \leq s \leq k-1$, such that

$$\{x_r, x_s\} \subseteq N(Y) \cap V(P) \subseteq \{x_r, \ldots, x_s\},$$

and

$$N(Z) \cap V(P) \setminus \{x_{r+1}, \ldots, x_{s-1}\} \neq \emptyset$$

for all components $Z$ of $G \setminus V(P)$. Note that $s \geq r+2$ as $G$ is 3-connected. Now contract $x_{r+1} \ldots x_{s-1}$ to a new vertex $x$ and call the resulting graph $G''$. Then $\|G''\| \leq 2|G''|-3$, and let $G_1 := G''[Y \cup \{x_r, x, x_s\}]$ and $G_2 := G'' \setminus Y$. As in Claim 13, we have

$$\|G_1\| + \|G_2\| = \|G''\| + 2 \leq 2|G''| - 1 = 2|G_1| + 2|G_2| - 7.$$  

Further, $x$ is neither a cut vertex of $G_1$ nor of $G_2$. Thus, by Corollary 3, either $G_1$ or $G_2$ has a stable cutset not containing $x$. But this is also a stable cutset of $G$, a contradiction. \qed
3 Complexity Issues

With \textsc{stable cutset} we mean the following decision problem: ‘Does a given graph admit a stable cutset?’ The computational complexity of \textsc{stable cutset} has been addressed in a number research papers, e.g., [5, 1, 7, 8, 9]. To sum up, \textsc{stable cutset} is NP-complete for graphs of maximum degree five (even for $K_4$-free planar graphs with maximum degree five [8] and for 5-regular line graphs of bipartite graphs [9]), and is trivial for graphs of maximum degree three (by Theorem 1, such graphs with more than seven vertices always have a stable cutset). The complexity status of \textsc{stable cutset} is still open for graphs with maximum degree four.

By Theorem 1, \textsc{stable cutset} for graphs with maximum degree four remains open only in four cases, namely for graphs with $n$ vertices and $m$ edges where $2n - 3 \leq m \leq 2n$. Thus, the following problem is of interest and has been addressed in [9, 10]:

\textsc{stable cutset}$(n, m)$. \textit{Given a graph $G$ with $n$ vertices and $m$ edges. Does $G$ have a stable cutset?}

It was shown in [9] that, for any given $\epsilon > 0$, \textsc{stable cutset}$(n, m)$ is NP-complete for $m \geq (2 + \epsilon)n$. By Theorem 1, \textsc{stable cutset}$(n, m)$ is trivial for $m \leq 2n - 4$. By Theorem 5, we obtain the following:

\textbf{Corollary 15.} \textsc{stable cutset}$(n, 2n - 3)$ is solvable in polynomial time.

\textit{Proof.} Let $G$ be a graph with $n$ vertices and $m = 2n - 3$ edges. Then, by Theorem 5, $G$ has a stable cutset, or else $G$ must belong to $\mathcal{G}_{sc}$. Since the members of $\mathcal{G}_{sc}$ can be recognized in time $O(n^4)$ in an obvious way, Corollary 15 follows.

In fact, the recognition of $G \in \mathcal{G}_{sc}$ can be performed in quadratic time, based on the following observations. For every edge $xy$, we can in linear time test if $\{x, y\}$ is a cutset and determine the components of $G - \{x, y\}$. If $G \in \mathcal{G}_{sc}$, performing this for all $2n - 3$ edges, this process yields in quadratic time a set of at most $n - 2$ components with at most a total of $3n - 6$ vertices, where each component is obtained from copies of $K_3$ and $\overline{C_6}$ via triangle identification. In particular, every vertex is in exactly one $K_3$. Further, every vertex of degree 2 is in a $K_3$-component, and the non-separating $K_3$ in other components consist exactly of the vertices of degree 3. This way, we can easily recover the building blocks, the $\overline{C_6}$, used to build up the components in quadratic time, by cutting off one $\overline{C_6}$ which includes a non-separating $K_3$ at a time (linear time for each step, linear number of steps).

With a bit more effort, one can show that one can decide if $G \in \mathcal{G}_{sc}$ in time $O(n \log n)$, but for the sake of exposition we do not present the argument here. \hfill \Box

\textbf{References}


