Eulerian circuits with no monochromatic transitions

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Abstract

Let $G$ be an eulerian digraph with a fixed edge coloring (not necessarily a proper edge coloring). A compatible circuit of $G$ is an eulerian circuit such that every two consecutive edges in the circuit have different colors. We characterize the existence of compatible circuits for directed graphs avoiding certain vertices of outdegree three. Our result is analogous to a result of Kotzig for compatible circuits in edge-colored eulerian undirected graphs.

From our characterization for digraphs we develop a polynomial time algorithm that determines the existence of a compatible circuit in an edge-colored eulerian digraph and produces a compatible circuit if one exists. Our results use the fact that rainbow spanning trees have been characterized in edge-colored undirected multigraphs. We provide another graph theoretical proof of this fact.

Keywords: Eulerian circuit, no monochromatic transitions, compatible circuit

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1 Introduction

A colored eulerian digraph is an eulerian digraph $G$ with a fixed edge coloring $\phi$ (where $\phi$ is not necessarily a proper edge coloring). A compatible circuit of $G$ is an eulerian circuit such that every two consecutive edges in the circuit have different colors. We prove necessary and sufficient conditions for the existence of a compatible circuit in colored eulerian digraphs that do not have certain vertices of outdegree three. The methods that we use give a polynomial time algorithm determining the existence of a compatible circuit and producing one if it exists.

Fleischner and Fulmek [10] provided sufficient conditions for the existence of a compatible circuit when the number of colors at each vertex is large. Isaak [14] gave stronger conditions for the existence of compatible circuits in digraphs, and used these results to show the
existence of certain universal cycles of permutations. In Section 3 we expand upon Isaak’s methods to determine when we can make local changes at a vertex to construct a compatible circuit. See Fleischner [9] for an overview of topics on eulerian digraphs, including compatible circuits in colored eulerian digraphs.

Kotzig [16] gave necessary and sufficient conditions for the existence of a compatible circuit in colored eulerian undirected graphs\(^1\). Our result for digraphs is analogous to Kotzig’s result. In an important application, Pevzner [22] used compatible circuits in undirected eulerian graphs to reconstruct DNA from its segments. Benkouar et al. [3] gave a polynomial time algorithm for finding a compatible circuit in a colored eulerian undirected graph, providing an alternate proof of Kotzig’s Theorem. They claimed that a similar algorithm holds for digraphs and gave a statement characterizing the existence of a compatible circuit in a colored eulerian digraph. However, their sufficient condition is false, as shown by the graph on the right in Figure 1.

There are many other results on finding subgraphs avoiding monochromatic transitions. Bollobás and Erdös [5] initiated the study of properly edge-colored hamiltonian cycles (which they called alternating hamiltonian cycles) in edge-colored complete graphs, and this study was continued in papers such as [1, 2]. The problem of finding properly edge-colored paths and circuits in edge-colored digraphs has been studied in several articles such as Gourvès et al. [12]; see the survey paper by Gutin and Jung Kim [13] for an overview. Finding subgraphs with all edges having different colors (called rainbow or heterochromatic) has also been well studied. Kano and Li [15] gave a survey paper on recent results about monochromatic and rainbow subgraphs in edge-colored graphs. In our methods for eulerian digraphs we use results on rainbow spanning trees in edge-colored undirected multigraphs.

This paper is organized as follows. In Section 2 we introduce definitions and elementary necessary conditions for the existence of a compatible circuit. Section 3 investigates when local changes at each vertex can be used to construct a compatible circuit. Section 4 contains our main result which characterizes when a colored eulerian digraph avoiding certain vertices of outdegree three has a compatible circuit. Section 5 discusses rainbow spanning trees in an edge-colored undirected multigraph. In Section 6 we present a polynomial time algorithm for finding a compatible circuit in a colored eulerian digraph. We conclude with some open questions in Section 7.

2 Preliminaries

In this section we introduce the basic definitions and describe elementary necessary conditions for the existence of a compatible circuit.

Throughout the paper we let $G$ be an eulerian digraph with a fixed edge coloring $\phi$ (where $\phi$ is not necessarily proper). We refer to $G$ as a colored eulerian digraph. A monochromatic transition is two consecutive edges in a walk of $G$ that have the same color. A compatible circuit $T$ is a an eulerian circuit with no monochromatic transitions. Our goal is to determine

\(^1\)Kotzig claimed this answered a question posed by Nash-Williams (see [10], page VI.1).
if a colored eulerian digraph has a compatible circuit. Though an eulerian circuit is defined to be a cyclic sequence of consecutive edges, it is determined by the transitions at each vertex of the digraph \( G \). We will focus on finding transitions at each vertex such that the resulting eulerian circuit is a compatible circuit.

Fleischner and Fulmek [10] considered a more general setting where the head and tail of each directed edge can receive different colors. Such graphs can be handled in our setting by subdividing edges with different colors on the head and tail. Similarly we can assume that a colored eulerian digraph is loopless, since subdividing the loop twice and coloring the new middle edge a new color results in a colored eulerian digraph with no loops. Henceforth we only consider loopless colored eulerian digraphs.

Now we establish some notation that will be used throughout the rest of the paper. Let \( G \) be a colored eulerian digraph and \( v \) a vertex of \( G \). Define \( E^+(v) \) to be the set of outgoing edges incident to \( v \) and \( E^-(v) \) to be the set of incoming edges incident to \( v \). For each vertex \( v \) define \( C_i(v) \) to be the set of incident edges to \( v \) that are colored with the color \( i \). We assume there are a total of \( k \) colors in the edge coloring. We refer to the sets \( C_1(v), \ldots, C_k(v) \) as the color classes of \( v \). Let \( \gamma(v) \) denote the size of the largest color class at \( v \). For each color \( i \), define \( C_i^+(v) \) to be the set of outgoing edges incident with \( v \) that are colored \( i \), and similarly \( C_i^-(v) \) for incoming edges.

An eulerian circuit \( T \) determines a matching between \( E^-(v) \) to \( E^+(v) \) corresponding to the transitions incident to a vertex \( v \). Hence, if \( \gamma(v) > \deg^+(v) \) for some vertex \( v \) then \( G \) does not have a compatible circuit; by the Pigeonhole Principle there will be a transition from an edge in \( E^-(v) \) to \( E^+(v) \) with the same color. Thus if \( G \) has a compatible circuit then \( \gamma(v) \leq \deg^+(v) \) for all vertices \( v \).

**Definition 2.1.** Let \( G \) be a colored eulerian digraph. Assume \( v \) is a vertex of \( G \) where \( \gamma(v) = \deg^+(v) \), and \( C_i(v) \) is a largest color class. If \( C_i(v) \) lies completely inside \( E^+(v) \) or \( E^-(v) \), then there are no restricted transitions at \( v \). Otherwise, if a compatible circuit exists it must match \( C_i^+(v) \) to \( E^-(v) \setminus C_i^-(v) \), and \( C_i^-(v) \) to \( E^+(v) \setminus C_i^+(v) \).

Construct a new colored eulerian digraph \( G' \) from \( G \), where for each vertex \( v \) with \( \gamma(v) = \deg^+(v) \) and the largest color class is not contained in \( E^+(v) \) or \( E^-(v) \), the vertex \( v \) is split into two new vertices \( v_1 \) and \( v_2 \), where \( v_1 \) has incoming edges \( E^-(v) \setminus C_i^-(v) \) and outgoing edges \( C_i^+(v) \), and \( v_2 \) has incoming edges \( C_i^-(v) \) and outgoing edges \( E^+(v) \setminus C_i^+(v) \) as depicted in Figure 1. The resulting digraph \( G' \) is reduced: a reduced colored eulerian digraph \( G \) is a loopless colored eulerian digraph with \( \gamma(v) \leq \deg^+(v) \) for all vertices \( v \in V(G) \) and if \( \gamma(v) = \deg^+(v) \) the largest color class is either all the incoming or all the outgoing edges.

**Lemma 2.2.** Let \( G \) be a colored eulerian digraph and \( G' \) be the reduced eulerian digraph created from \( G \). Then \( G \) has a compatible circuit if and only if \( G' \) has a compatible circuit.

**Proof.** A compatible circuit \( T \) of \( G' \) viewed as a series of edges is also a compatible circuit in \( G \). If \( G \) has a compatible circuit \( T \), then for each vertex \( v \) with \( \gamma(v) = \deg^+(v) \) the edges in \( C_i^+(v) \) must be matched to \( E^-(v) \setminus C_i^-(v) \) and the edges in \( C_i^-(v) \) must be matched to \( E^+(v) \setminus C_i^+(v) \). So \( T \) is also a compatible circuit in \( G' \). \qed
Figure 1: The vertex $v$ is replaced with the vertices $v_1$ and $v_2$ when $\gamma(v) = \text{deg}^+(v)$ and the largest color class has both incoming and outgoing edges. The example on the the right is a digraph with no compatible circuit.

Lemma 2.2 shows the equivalence between the colored eulerian digraph $G$ and the reduced eulerian digraph $G'$. Hence, throughout the rest of the paper we only consider reduced eulerian digraphs.

3 Fixable vertices

In the case when $\gamma(v) = \text{deg}^+(v)$, if the largest color class is contained in $E^+(v)$ or $E^-(v)$ then there are no restricted transitions. Thus, every eulerian circuit has no monochromatic transitions between $E^-(v)$ and $E^+(v)$. In this section we investigate when we can change the transitions of an eulerian circuit at a single vertex $v$ to create a new eulerian circuit with no monochromatic transitions at $v$.

Definition 3.1. An eulerian circuit $T$ determines a matching between $E^+(v)$ and $E^-(v)$ by considering the segments $S_1, \ldots, S_d$ between successive appearances of $v$ in $T$. We refer to these segments $S_1, \ldots, S_d$ as excursions of $T$. There is a natural matching between $E^+(v)$ and $E^-(v)$ where the first edge of $S_i$ is matched to the last edge of $S_i$. We wish to find nonmonochromatic transitions at $v$ such that the excursions are combined into one circuit.

Since it is not immediate which matchings of $E^+(v)$ and $E^-(v)$ arise from the excursions of some eulerian circuit, we consider any matching $M$ between $E^+(v)$ and $E^-(v)$. Label the edges incident to $v$ as $e_1^-, e_1^+, \ldots, e_d^-, e_d^+$, where $d = \text{deg}^+(v)$ and $e_i^- \in E^-(v)$ is matched in $M$ to $e_i^+ \in E^+(v)$ for $i = 1, \ldots, d$. The excursion graph $L_M(v)$ is the colored digraph with vertex set consisting of $v$ and the disjoint union of $N^-(v)$ and $N^+(v)$. The edge set of $L_M(v)$ consists of all edges in $G$ incident to $v$, along with edges from $e_i^+(v)$ to $e_i^-(v)$ for all $i$. The edges incident to $v$ retain their color from $G$ and the new edges receive a new color $k + 1$ not in $G$. Note that the excursion graph $L_M(v)$ is a colored eulerian digraph consisting of cycles containing $v$ of length three; see the right graph in Figure 2 for an example. When the matching $M$ arises from an eulerian circuit $T$, we write $L_T(v)$ to denote the excursion graph $L_M(v)$.
Definition 3.2. Let $G$ be a colored eulerian digraph. A vertex $v$ is fixable if $L_M(v)$ has a compatible circuit for every matching $M$ between $E^+(v)$ and $E^-(v)$.

The usefulness of fixable vertices is clear from the following proposition.

Proposition 3.3. Let $G$ be a colored eulerian digraph. If all the vertices of $G$ are fixable then $G$ has a compatible circuit.

Proof. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of $G$. Let $T_0$ be an arbitrary eulerian circuit of $G$. Since $v_1$ is fixable, the excursion graph $L_{T_0}(v_1)$ has a compatible circuit $W_0$. The circuit $W_0$ determines a set of transitions between $E^-(v_1)$ and $E^+(v_1)$. Let $d_1 = \deg^+(v_1)$. We use these transitions to alter the circuit $T_0$ by rearranging the order in which the excursions $S_1, \ldots, S_d$ occur according to the transitions found in the compatible circuit. Since $L_{T_0}(v_1)$ is compatible the resulting trail is an eulerian trail of $G$ which we call $T_1$. The eulerian trail $T_1$ has no monochromatic transitions at $v_1$ and introduces no new monochromatic transitions at other vertices. For each $i = 2, \ldots, n$ we repeat the previous process with the excursion graph $L_{T_{i-1}}(v_i)$. We obtain a sequence of eulerian circuits $T_0, T_1, \ldots, T_n$ where $T_i$ has no monochromatic transitions at $v_j$ for $1 \leq j \leq i$. Thus, $T_n$ has no monochromatic transitions and hence is a compatible circuit of $G$. \qed

From this proposition we see that reduced colored eulerian digraphs that do not have compatible circuits must have nonfixable vertices. In the rest of this section we characterize fixable vertices. We use the same approach taken by Isaak [14] to find compatible circuits. Our proof uses Meyniel’s Theorem [19] rather than Ghouila-Houri’s Theorem [11], which gives a slightly stronger result.

Theorem 3.4 (Meyniel [19]). Let $G$ be a digraph on $n$ vertices with no loops. If

$$
\deg^+(x) + \deg^-(x) + \deg^+(y) + \deg^-(y) \geq 2n - 1
$$

for all pairs of nonadjacent vertices $x$ and $y$ in $G$, then $G$ is hamiltonian.

As a direct consequence of Meyniel’s Theorem we have the following proposition.

Proposition 3.5. Let $G$ be a colored eulerian digraph with a vertex $v$. If $\gamma(v) \leq \deg^+(v) - 2$ or $\gamma(v) = \deg^+(v) - 1$ and the second largest color class has size strictly smaller than $\deg^+(v) - 1$, then $v$ is fixable.

Proof. Consider the excursion graph $L_M(v)$ for an arbitrary matching $M$ between $E^+(v)$ and $E^-(v)$. Let $S_1, \ldots, S_d$ denote the directed 3-cycles of $L_M(v)$, where $d = \deg^+(v)$.

Create a new digraph $D$ with vertex set $S_1, \ldots, S_d$ where there is a directed edge from $S_i$ to $S_j$ if $S_j$ can follow $S_i$ in a compatible circuit, i.e. $i \neq j$ and $e_i^-(v)$ is not the same color as $e_j^+(v)$.

By construction, a hamiltonian cycle of $D$ corresponds to a compatible circuit of $L_M(v)$. We apply Meyniel’s Theorem to show $D$ is hamiltonian.

Assume that $S_i$ and $S_j$ are distinct nonadjacent vertices in $D$. Then $e_i^-(v)$ and $e_j^+(v)$ have the same color, and $e_i^+(v)$ and $e_j^-(v)$ have the same color. Without loss of generality
assume that \( e_i^- (v) \) has color 1 and \( e_i^+ (v) \) has color 2. The outdegree of \( S_i \) in \( D \) is at least \( d - |C_i^+(v)| - 1 \), since \( e_i^- (v) \) has color 1 and there are \( |C_i^+(v)| \) edges with color 1 on the other 3-cycles and there is no loop at vertex \( S_i \) in \( D \). Similarly the indegree of \( S_i \) is at least \( d - |C_i^-(v)| - 1 \), the outdegree of \( S_j \) is at least \( d - |C_j^+(v)| - 1 \) and the indegree of \( S_j \) is at least \( d - |C_j^-(v)| - 1 \). Therefore, the sum of the indegree and outdegree of the vertices \( S_i \) and \( S_j \) in \( D \) is at least
\[
4d - (|C_i^+(v)| + |C_i^-(v)| + |C_j^+(v)| + |C_j^-(v)|) - 4 = 4d - |C_i^+(v)| - |C_j^-(v)| - 4.
\]
Without loss of generality assume that the size of \( C_i(v) \) is at least as large as the size of \( C_j(v) \). By hypothesis \( |C_i(v)| \leq \deg^+(v) - 1 \) and \( |C_j(v)| \leq \deg^+(v) - 2 \). Therefore the sum of the degrees of \( S_i \) and \( S_j \) in \( D \) is
\[
\deg^-(S_i) + \deg^-(S_i) + \deg^-(S_j) + \deg^-(S_j) \geq 4d - (2d - 1) - (2d - 2) - 4 = 2d - 1.
\]
By Meyniel’s Theorem, \( D \) has a hamiltonian cycle, so \( L_M(v) \) contains a compatible circuit. Since the choice of \( M \) was arbitrary, \( v \) is a fixable vertex.

As a consequence of Proposition 3.5, if each vertex of a colored eulerian digraph has enough color classes then the graph has a compatible circuit.

**Corollary 3.6.** Let \( G \) be a reduced colored eulerian digraph. If each vertex has at least five different color classes then \( G \) has a compatible circuit.

**Proof.** If each vertex \( v \) has at least five different color classes and \( \gamma(v) \leq \deg^+(v) - 1 \), then the second largest color class has size at most \( \deg^+(v) - 2 \). By Proposition 3.5 all the vertices where \( \gamma(v) < \deg^+(v) \) are fixable, and since \( G \) is reduced all the vertices with \( \gamma(v) = \deg^+(v) \) are fixable. By Proposition 3.3 \( G \) has a compatible circuit.

Notice that Meyniel’s Theorem does not apply when \( C_x(v) \) and \( C_y(v) \) are both of size \( \gamma(v) = \deg^+(v) - 1 \). The proposition will provide a characterization of fixable vertices.

**Proposition 3.7.** Let \( G \) be a colored eulerian digraph and \( v \) a vertex of \( G \) where \( \gamma(v) = \deg^+(v) - 1 \) and \( \deg^+(v) \geq 2 \). Then the graph \( L_M(v) \) has a compatible circuit, unless the following properties hold:

1. The two largest color classes, \( C_x(v) \) and \( C_y(v) \), are of size \( \deg^+(v) - 1 \).
2. The color classes \( C_x(v) \) and \( C_y(v) \) have both incoming and outgoing edges (i.e. the sets \( C_x^-(v), C_x^+(v), C_y^-(v) \) and \( C_y^+(v) \) are all nonempty).
3. The matching \( M \) matches \( C_x^-(v) \) to \( C_y^+(v) \); \( C_x^+(v) \) to \( C_y^-(v) \); and the two edges not in \( C_x(v) \) and \( C_y(v) \) are matched together (hence one is an incoming edge and the other is an outgoing edge).

If the above properties hold then \( L_M(v) \) does not have a compatible circuit.
Figure 2 illustrates the properties of the proposition.

**Proof.** First we show that the colored eulerian digraph $L_M(v)$ satisfying the above properties does not have a compatible circuit. Notice that the 3-cycles where $C_x^{-}(v)$ is matched to $C_y^{+}(v)$ can not transition to the 3-cycles where $C_y^{-}(v)$ is matched to $C_x^{+}(v)$. The only other 3-cycle in the excursion graph is created by the two edges not in the largest color class. This 3-cycle can transition from the 3-cycles where $C_x^{-}(v)$ is matched to $C_y^{+}(v)$ (or vice versa) but it can not be used to transition back. So in every eulerian circuit in $L_M(v)$ there is a monochromatic transition at $v$.

We prove the vertices not satisfying the above properties are fixable by induction on $d = \deg^{+}(v)$. Proposition 3.5 handles the case when the second largest color class is of size strictly less than $\deg^{+}(v) - 1$. The base cases are when $d = 2, 3, 4$. The case for $d = 2$ is an outdegree two vertex where each edge has a distinct color. This vertex has no restrictions so it is fixable. For $d = 3, 4$ we used Sage [24] to examine all possible color combinations in the case we have two color classes of size $\deg^{+}(v) - 1$. We colored the six (or eight) edges and checked all possible hamiltonian cycles on three (or four) vertices to find a compatible circuit. We found a compatible circuit except in the cases where the properties in the proposition hold. This required checking 180 configurations for $d = 3$ and 1120 configurations for $d = 4$.

Assume that $d > 4$. Pick an edge $e_1$ in $C_x(v)$ to match with an edge $e_2$ in $C_y(v)$. Then fixing this transition and splitting the transition off the digraph of $L_M(v)$ creates a new digraph $L'_M(v)$ which only has $d - 1$ directed 3-cycles. We will pick this transition so we can apply induction to the new colored digraph $L'_M(v)$.

Let $C_s(v)$ denote the two edges incident to $v$ not colored $x$ or $y$. Suppose that the two edges in $C_s(v)$ are matched together, i.e. $C_s(v) = \{e_i^{-}(v), e_i^{+}(v)\}$ for some $i$. If the properties above do not hold, then either $C_x^{+}(v) = \emptyset$, $C_x^{-}(v) = \emptyset$, or there exists $j$ such that $e_j^{-}(v)$ and $e_j^{+}(v)$ both belong to $C_x(v)$.

First consider the case when $C_x^{+}(v) = \emptyset$; the case when $C_x^{-}(v) = \emptyset$ uses a symmetric argument. Pick an edge $e_{j_k}^{-}(v)$ in $C_x(v) = C_x^{-}(v)$ and match it with an edge $e_{j_k}^{+}(v)$ in $C_y(v) = C_y^{+}(v)$ where $j \neq k$. Fix this transition and split those two edges off the vertex $v$. This operation combines the 3-cycles $j$ and $k$ to form one cycle with outgoing edge $e_k^{+}(v)$ and incoming edge $e_{j_k}^{-}(v)$. Contract and recolor the appropriate edges to form a new excursion.
graph $L'_M(v)$. The new digraph still avoids the matching described above, and hence we can apply induction to $L'_M(v)$.

Now consider the case when there exists $j$ such that $e^j_-(v)$ and $e^j_+(v)$ both belong to $C_x(v)$. If $e^-_k(v)$ and $e^+_k(v)$ have the same color for $k = 1, \ldots , i - 1, i + 1, \ldots , d$, then fix a transition $e^-_k(v)$ and $e^+_k(v)$ where $e^+_k(v)$ is colored $y$. Splitting this transition off results in a new excursion graph avoiding the matching described above, and hence we can apply induction. Otherwise there exists $k \neq i, j$ with either $e^+_k(v) \in C_x(v)$ and $e^-_k(v) \in C_y(v)$, or $e^-_k(v) \in C_x(v)$ and $e^+_k(v) \in C_y(v)$. If $e^-_k(v) \in C_y(v)$ match $e^-_k(v)$ to $e^+_j(v)$ and if $e^-_k(v) \in C_y(v)$ match $e^-_j(v)$ to $e^+_k(v)$. After splitting off this transition we can apply induction.

We now consider the case when the edges of $C_x(v)$ are not matched together. Without loss of generality we assume that $C_x(v) \cap E^+(v) \neq \emptyset$; otherwise we can use the symmetric argument by switching $+$ and $−$. Let $e^+_j(v) \in C_x(v)$ and assume the other edge of $C_x(v)$ is either $e^+_j(v)$ or $e^-_j(v)$. Without loss of generality assume that $e^-_j(v) \in C_x(v)$ (the same argument will work if $e^-_j(v) \in C_x(v)$ by switching $x$ and $y$). If $C^+_y(v) \setminus \{e^+_j(v)\} \neq \emptyset$ then we can match $e^-_j(v)$ to an edge from this set and we will still have the two edges in $C$ not matched. Therefore after splitting off this transition we can apply induction.

Assume that $C^+_y(v) \setminus \{e^+_j(v)\} = \emptyset$. Then $C_x(v) = C^+_x(v) \cup e^-_j(v)$, where $C^+_x(v) = E^+(v) \setminus \{e^+_j(v), e^-_j(v)\}$. Since $d > 4$ we know that $|C^+_x(v)| > 2$ and $|C^+_y(v) \setminus \{e^-_j(v)\}| > 2$, therefore we can match an edge from $C^+_x(v)$ to an edge in $C^-_y(v) \setminus \{e^-_j(v)\}$. After splitting off this transition we can apply induction.

**Corollary 3.8.** Let $G$ be a colored eulerian digraph. A vertex $v$ is fixable if and only if $v$ satisfies one of the following:

1. $γ(v) ≤ deg^+(v) - 2$.
2. $γ(v) = deg^+(v) - 1$ and the second largest color class has size strictly smaller than $deg^+(v) - 1$.
3. $γ(v) = deg^+(v) - 1$ and there are two color classes of size $γ(v) = deg^+(v) - 1$, where the two edges not in the largest two color classes are both incoming or both outgoing edges.
4. $γ(v) = deg^+(v) - 1$ and there are two color classes of size $γ(v) = deg^+(v) - 1$, where one of the largest color classes has only incoming edges or only outgoing edges.
5. $γ(v) = deg^+(v)$ where a largest color class $C_i(v)$ has only incoming edges or only outgoing edges.

**Proof.** By Proposition 3.5, Proposition 3.7, and the discussion of vertices $v$ with $γ(v) = deg^+(v)$ we know the above vertices are fixable.

In the case when $γ(v) > deg^+(v)$ the excursion graph $L_M(v)$ never has a compatible circuit. So all vertices $v$ with $γ(v) > deg^+(v)$ are not fixable. If $γ(v) = deg^+(v)$ and if a largest color class $C_i(v)$ has both incoming and outgoing edges then we can create a matching $M$ where $C_i^+(v)$ is matched to $E^-(v) \setminus C^-_i(v)$. When this happens the excursion graph $L_M(v)$ does not have compatible circuit since the 3-cycles with an edge from $C_i^+(v)$ can never be matched with the 3-cycles from $C^-_i(v)$.

Proposition 3.7 shows that the only nonfixable vertices with $γ(v) = deg^+(v) - 1$ are those with two color classes of size $deg^+(v) - 1$ and the two other edges have an incoming and
Proposition 3.9. Let $G$ be a colored eulerian digraph, $v$ a fixable vertex of $G$, and $T$ a (not necessarily compatible) eulerian circuit of $G$. Then there exists a polynomial time algorithm to find a compatible circuit in $L_T(v)$.

Proof. Berman and Liu [4] gave a polynomial time algorithm to find a hamiltonian cycle in a digraph that satisfies the hypothesis of Meyniel’s Theorem. Hence for a fixable vertex $v$ that satisfies the hypotheses in Proposition 3.5 we can find in polynomial time a hamiltonian cycle in $D$, and thus a compatible circuit in $L_T(v)$.

In the case when $\gamma(v) = \deg^+(v) - 1$ the proof of Theorem 3.7 gives a polynomial time algorithm: we iteratively match edges from the largest color classes together until we have an excursion graph with only four excursions.

4 Graphs that contain nonfixable vertices

In this section we examine reduced colored eulerian digraphs with nonfixable vertices. Let $G$ be a reduced eulerian digraph and let $S$ be the set of nonfixable vertices in $G$. The set $S$ consists of the vertices described in Lemma 3.7; these vertices have two color classes of size $\gamma(v) = \deg^+(v) - 1$ where both largest color classes have incoming and outgoing edges and there is one incoming and one outgoing edge not in the largest two color classes. Let $S_3$ be the nonfixable vertices $v$ with $\deg^+(v) = 3$ with exactly three color classes.

For the rest of this section we will assume that $S_3 = \emptyset$. First, we will describe several related graphs that model the important properties of colored eulerian digraphs when $S_3 = \emptyset$.

Definition 4.1. Let $G$ be a reduced colored eulerian digraph and $S$ be the set of nonfixable vertices. For each vertex $v \in S$, let $C_x(v)$ and $C_y(v)$ denote the two largest color classes, and define $C_*(v) = \{e^+(v), e^-(v)\}$ to be the two edges incident to $v$ not in $C_x(v) \cup C_y(v)$. Note that since $S_3 = \emptyset$ the set $C_*(v)$ is well defined.

We construct a new colored digraph $G_S$ by splitting all vertices in $S$ as follows: for $v \in S$ replace $v$ with three new vertices $v_1, v_2$ and $v_3$ where

- $v_1$ is incident to the edges in $C_*(v)$,
- $v_2$ is incident to the edges in $C_x^-(v) \cup C_y^+(v)$, and
- $v_3$ is incident to the edges in $C_x^+(v) \cup C_y^-(v)$.

Notice that when the two edges in $C_*(v)$ are deleted the resulting vertex $v'$ has $\gamma(v') = \deg^+(v')$ and using the splitting from Definition 2.1 on $v'$ results in the creation of the vertices $v_2$ and $v_3$.

Definition 4.2. Define the component graph $H_G$ of $G_S$ as follows: the vertices of $H_G$ are the strong components of $G_S$ (note that since $\deg^+(v) = \deg^-(v)$ for all vertices in $G_S$ the strong components are also the weak components). For each vertex $v \in S$ there is an edge in
$H_G$ labeled with $v$ between $D_1$ and $D_2$ where the component $D_1$ contains $v_1$ and $D_2$ contains $v_2$, and another edge in $H_G$ labeled $v$ between $D_1$ and $D_3$ where the component $D_1$ contains $v_1$ and $D_3$ contains $v_3$.

Note that the component graph $H_G$ is an undirected edge labeled multigraph. See Figures 3 and 6 for pictures of the graphs $G$, $G_S$, and $H_G$.

![Diagram](image.png)

Figure 3: An example of a colored eulerian digraph with two nonfixable vertices and the auxiliary graphs $G_S$ and $H_G$.

**Definition 4.3.** A 2-trail is a set of two incident edges in an undirected graph. Either of the edges may be a loop, or the edges may be multiple edges.

Notice that in $H_G$ the edges with label $v \in S$ form a 2-trail, and so the edge set of $H_G$ can be thought of as a union of 2-trails. As we will see in Theorem 4.5 there is a compatible circuit in $G$ if and only if there exists an appropriate connected subgraph of $H_G$. We will need the following definition to make this precise.

**Definition 4.4.** A 2-trail traversal is a set $E'$ of edges in $H_G$ with exactly one edge from each 2-trail such that the spanning subgraph of $H_G$ with edge set $E'$ is connected. Let $H'_G$ be the connected spanning subgraph of $H_G$ with edge set $E'$.

**Theorem 4.5.** Let $G$ be a reduced colored eulerian digraph with no nonfixable vertices of outdegree three with exactly three color classes (i.e. $S_3 = \emptyset$). Then the graph $G$ has a compatible circuit if and only if the component graph $H_G$ has a 2-trail traversal.

**Proof.** ($\Rightarrow$) Let $T$ be a compatible circuit of $G$. From $T$ we will construct a 2-trail traversal $E'$ of $H_G$.

For each vertex $v \in S$, let $C_x(v)$ and $C_y(v)$ be the two largest color classes at $v$, and let $C_*(v) = \{e^-(v), e^+(v)\}$ denote the two edges not in $C_x(v) \cup C_y(v)$. In $T$ the two edges in $C_*(v)$ satisfy exactly one of the three following:
1. the two edges in \( C_s(v) \) are matched together, or
2. \( e^- (v) \) is matched to an edge in \( C_s^+(v) \) and \( e^+ (v) \) is matched to an edge in \( C_s^-(v) \), or
3. \( e^- (v) \) is matched to an edge in \( C_{s_1}^+(v) \) and \( e^+ (v) \) is matched to an edge in \( C_{s_2}^-(v) \).

\( T \) is a cyclic sequence of edges in \( G \), so it is a sequence of edges in \( G_s \). Since \( T \) is an eulerian circuit, the sequence visits each edge exactly once in \( G \) and in \( G_s \). Let \( v \) be a nonfixable vertex of \( G \) and \( e^- (v) \) be the incoming edge incident to \( v_1 \) in \( G_s \). Let \( e \) be the edge following \( e^- (v) \) in \( T \).

If the edge \( e \) is in the same component in \( G_s \) as \( e^- (v) \), then \( e \) is \( e^+ (v) \). Let \( f_v \) be an arbitrary edge in \( H_G \) labeled \( v \). If the edge \( e \) is in a different component in \( G_s \) than \( e^- (v) \), then \( e \) has as its tail the vertex \( v_2 \) or \( v_3 \). There is an edge \( f_v \) labeled \( v \) in \( H_G \) that corresponds to the vertices \( v_1 \) and the tail of the edge \( e \). Notice that the two edges in \( C_s(v) \) satisfy (2) or (3) above. The edge \( e' \) preceding \( e^+ (v) \) in \( T \) is such that the head of \( e' \) is the tail of \( e \) in \( G_s \); hence \( f_v \) is consistent for the edges \( e^- (v) \) and \( e \), and \( e' \) and \( e^+ (v) \).

Let \( E' = \{ f_v : v \in S \} \) denote the set of edges \( f_v \) corresponding to the nonfixable vertices as described above. By construction there is exactly one edge from each 2-trail in \( E' \). We now show the spanning subgraph of \( H_G \) with edge set \( E' \) is connected. Let \( D_1 \) and \( D_2 \) be two vertices of \( H_G \). Since \( G_s \) has no isolated vertices we can pick an edge \( e_1 \) in the component \( D_1 \) in \( G_s \) and an edge \( e_2 \) in the component \( D_2 \) in \( G_s \). Start at the edge \( e_1 \) in the circuit \( T \) and follow the circuit until we reach the edge \( e_2 \). As we move along the sequence \( T \), whenever we move from an edge in one component in \( G_s \) to an edge in another component, there is a corresponding edge in \( E' \) between these components in \( H_G \). Thus, following the circuit in \( G_s \) from \( e_1 \) to \( e_2 \) induces a walk in \( H'_G \) from \( D_1 \) to \( D_2 \). Hence \( H'_G \) is connected, and so \( E' \) is a 2-trail traversal.

\((\Leftarrow)\) Now assume that \( H_G \) has a 2-trail traversal \( E' \). Recall that \( H'_G \) is the spanning subgraph of \( H_G \) with edge set \( E' \). We will form a new graph \( G' \) from \( G_s \) by identifying pairs of vertices according to the 2-trail traversal. The operation of identifying two vertices \( v_1 \) and \( v_i \) is to remove \( v_1 \) and \( v_i \) from the graph and create a new vertex \( v' \) that is incident to the disjoint union of the edges incident to \( v_1 \) and \( v_i \).

First we assume that \( \deg^+(v_2) \geq 2 \) and \( \deg^+(v_3) \geq 2 \) in \( G_s \) for all nonfixable vertices \( v \in S \). Each edge in the 2-trail traversal corresponds to two vertices \( v_1 \) and \( v_i \) for \( i \in \{2, 3\} \) in \( G_s \). For all edges \( e \) in \( E' \), identify the corresponding vertex \( v_1 \) with the vertex \( v_i \). Call the resulting graph \( G' \).

The graph \( G' \) is a colored digraph with \( \deg^+(v) = \deg^-(v) \) for all vertices \( v \in V(G') \). Since \( \deg^+(v_1) \geq 2 \) the vertex in \( G' \) created by identifying \( v_1 \) and \( v_i \) is a fixable vertex by Corollary 3.8, so all vertices of \( G' \) are fixable. We claim that \( G' \) is connected. Let \( r' \) and \( s' \) be two vertices in \( G' \). If \( r' \) was created by identifying vertices \( v_1 \) and \( v_i \), arbitrarily pick \( v_1 \) or \( v_i \) and call it \( r \). If not, then \( r' \) is also a vertex in \( G_s \), but we write \( r \) for the vertex \( r' \) in \( G_s \). Let \( D_1 \) be the component in \( G_s \) containing \( r \) and \( \hat{D} \) be the component containing \( s \). Since \( E' \) is a 2-trail traversal there is a walk \( W = D_1 D_2 \ldots D_k = \hat{D} \) in \( H_G \) using edges of \( E' \). Since \( D_i D_{i+1} \) is an edge in \( E' \) there exist vertices \( x_i \in D_i \) and \( y_{i+1} \in D_{i+1} \) identified together in \( G' \); call this vertex \( z_i \). Since \( y_i \) and \( x_i \) are in the same component \( D_i \) in \( G_s \) there exists a walk from \( y_i \) to \( x_i \) and hence there is a walk from \( z_{i-1} \) to \( z_i \). There are also walks...
Figure 4: The identification of $v_1$ and $v_i$ to form $G'$ and splitting to create $\hat{G}'$.

from $x$ to $z_1$ and $z_{k-1}$ to $y$ in $G'$. Concatenating these walks forms a walk from $r$ to $s$ in $G'$. Thus $G'$ is connected.

By Proposition 3.3, $G'$ has a compatible circuit $T$. Since incident edges in $G'$ are incident in $G$, the circuit $T$ is a compatible circuit of $G$.

In the case when $\deg^+(v_2)$ or $\deg^+(v_3)$ has outdegree 1 we need to be more careful. In the case when the vertex $v_1$ is identified with the vertex $v_i$ of outdegree 1 it does not form a fixable vertex in $G'$ if the edges in $C_*(v)$ have the same color. Instead the created vertex requires an additional split as in Definition 2.1, which could disconnect $G'$.

We claim that if $H_G$ has a 2-trail traversal then it has a 2-trail traversal $E'$ with the property that for any edge $e \in E'$ where
- $e$ corresponds to vertices $v_1$ and $v_i$, where $\deg^+(v_i) = 1$ in $G_S$, and
- both edges of $C_*(v)$ have the same color
then $e$ is a bridge in the spanning subgraph of $H_G$ with edge set $E'$.

By hypothesis there exists a 2-trail traversal. Both of the edges labeled $v$ satisfy the two properties above only when $v \in S_3$. Since $S_3$ is empty, if $E'$ has an edge $e$ labeled $v$ with the properties above then we may replace the edge $e$ with the other edge $f$ labeled $v$, thereby reducing the number of edges in $E'$ with the above properties.

As before, we form the graph $G'$ by identifying vertices in $G_S$ according to the edges in $E'$. Identified vertices in $G'$ are fixable unless $v_1$ and $v_i$ both have degree 1 in $G_S$ and the two edges in $C_*(v)$ have the same color. In this case, the edge labeled $v$ in $H_G$ in $E'$ is a bridge in $H'_G$, so the identified vertex $v'$ in $G'$ is a cut vertex in the graph $G'$.

Let $e^-(v)$ and $e^+(v)$ be the incoming and outgoing edges of $v_1$ and $e'$ and $e$ be the incoming and outgoing edges of $v_i$. Then we split the vertex $v'$ according to Definition ??.

Replace the vertex $v'$ with two new vertices $v'_1$ and $v'_2$ where $v'_1$ has incoming edge $e^-(v)$ and outgoing edge $e$ and $v'_2$ has incoming edge $e'$ and outgoing edge $e^+(v)$. Note that the split at $v'$ preserves connectivity since $v'$ is a cut vertex, as shown in Figure 4. Note that the condition $v'$ is a cut vertex is sufficient as shown in Figure 5.
Applying this splitting to all vertices $v'$ in $G'$ where $v_1$ and $v_i$ both have degree 1 in $G_S$ and the two edges in $C_*(v)$ have the same color results in a connected eulerian digraph $\hat{G}'$ where $\deg^+(v) = \deg^-(v)$ for all vertices $v \in V(\hat{G}')$ and where all the vertices are fixable. Again by Proposition 3.3, $\hat{G}'$ has a compatible circuit $T$, and since incident edges in $\hat{G}'$ are incident in $G$ the circuit $T$ is a compatible circuit of $G$.

Figure 3 shows a colored eulerian digraph with no compatible circuit since $H_G$ does not have a 2-trail traversal. Figure 6 gives an example of a colored eulerian digraph with a compatible circuit and the associated graphs $G_S$, $H_G$, $G'$, and $\hat{G}'$.

Theorem 4.5 provides necessary and sufficient conditions for the existence of a compatible circuit in reduced colored eulerian graphs where $S_3 = \emptyset$. The usefulness of auxiliary graphs similar to $G_S$ and $H_G$ in graphs with $S_3 \neq \emptyset$ is unclear. We can define an auxiliary graph $G_S$ where each vertex $v \in S_3$ is replaced with six vertices where each vertex is adjacent to a distinct edge of $v$. Between these six vertices in $G_S$ there are two matchings that connect the six vertices together. Again we have a similar result that $G$ has a compatible circuit if and only if we can pick one edge from each 2-trail for vertices $v \notin S_3$ and one of the two matchings for each vertex $v \in S_3$. We show in the next section there is a nice characterization of finding 2-trail traversals since they are closely related to rainbow spanning trees of undirected multigraphs. However, it is unclear if there is a good characterization for this other model.

5 Rainbow spanning trees

In this section we discuss necessary and sufficient conditions for the existence of a 2-trail traversal in the graph $H_G$.

**Definition 5.1.** Let $H$ be an undirected multigraph with a fixed edge coloring (not necessarily proper). A *rainbow spanning tree* is a spanning tree of $H$ that has at most one edge from each color class.
A 2-trail traversal for the graph $H_G$ contains a rainbow spanning tree where the color classes are the 2-trails. Conversely, a rainbow spanning tree can be extended to a 2-trail traversal by adding edges. Therefore the problem of determining if a 2-trail traversal exists in $H_G$ is equivalent to the problem of finding a rainbow spanning tree.

Broersma and Li [6] showed that determining the largest rainbow spanning forest of $H$ can be solved by applying the Matroid Intersection Theorem [7] (see Schrijver [23], page 700) to the graphic matroid and the partition matroid on the edge set of $H$ defined by the color classes. Schrijver [23] translated the conditions of the Matroid Intersection Theorem into necessary and sufficient conditions for the existence of a rainbow spanning tree, stated below in Theorem 5.2. Suzuki [25] gave a graph-theoretical proof of the same theorem.

**Theorem 5.2.** A graph $H$ has a rainbow spanning tree if and only if for every partition $\pi$ of $V(H)$, the number of color classes between the parts of $\pi$ is at least $\#(\pi) - 1$, where $\#(\pi)$ is the number of parts in $\pi$.

Theorem 5.2 is similar to a result by Tutte [26] and Nash-Williams [21] on finding $k$-edge disjoint spanning trees. For completeness we provide a graph-theoretical proof of Theorem 5.2.

**Proof.** For any partition $\pi$ of $V(H)$ a spanning tree must have at least $\#(\pi) - 1$ edges between the partitions, since a tree is connected. Thus, there must be at least $\#(\pi) - 1$
distinct colored edges between the parts of $\pi$ in a rainbow spanning tree, and hence in $H$ as well.

To prove sufficiency, assume that for every partition $\pi$ of $V(H)$ the number of color classes between parts is at least $\#(\pi) - 1$. Let $1, \ldots, k$ be the $k$ color classes. For any subset $S$ of the edges of $H$, let $\chi(S, i)$ denote the number of edges in $S$ with the color $i$, and let $\sigma(S) = \sum_{i=1}^{k} \chi(S, i) - 1$. When viewing a spanning tree $T$ as a set of edges, then $\sigma(T) \geq 0$ for all spanning trees, with equality if and only if $T$ is a rainbow spanning tree.

Let $T$ be a spanning tree of $H$ that minimizes the value $\sigma(T)$ over all spanning trees. We want to show that $\sigma(T) = 0$. Assume that $\sigma(T) > 0$. Then there exists at least two edges in the tree $T$ with the same color $c$. Label all the edges of $T$ with color $c$ with $a_1$. We inductively extend the edge labeling of $T$ in the following way: suppose the labels $a_1, \ldots, a_{i-1}$ have been assigned. An unlabeled edge $e$ of $T$ is labeled $a_i$ if there exists an edge $e'$ in $T$ labeled $a_{i-1}$ and an edge $f$ of $H$ that has the same color as $e$ such that $T + f - e'$ is a spanning tree of $H$. The process terminates when no such unlabeled edge exists. Note the process may terminate leaving some edges of $T$ unlabeled.

If there is an edge $e_i$ with label $a_i$ in $T$ and an edge $f$ in $H$ where $T + f - e_i$ is a tree, where there are no edges of $T$ with the same color as $f$, then we can create another spanning tree with smaller $\sigma$ value than $T$. Add $f$ to the tree $T$ and delete the edge $e_i$ to form the tree $T_i = T + f - e_i$. Since $e_i$ was labeled with $a_i$ there exists an edge $e_{i-1}$ of $T$ with label $a_{i-1}$ and an edge $f_{i-1}$ that has the same color as $e_i$. Add $f_{i-1}$ and delete $e_{i-1}$ to form the tree $T_{i-1}$. Continuing this process, we obtain the tree $T_1$ where $T_1 = T + f - e_i + f_{i-1} - e_{i-1} + \cdots - e_1 = T + f + (-e_i + f_{i-1}) + \cdots + (-e_2 - f_1) - e_1$. Note that $e_i$ and $f_{i-1}$ have the same color for $i \geq 2$ and adding $f$ and removing $e_1$ gives us that $\sigma(T_1) = \sigma(T) - 1 < \sigma(T)$. By the extremal choice of $T$ such an edge $f$ can not exist.

Therefore, for every edge $e_i$ with label $a_i$ all the edges $f$ in $H$ who lie between the components of $T - e_i$ have the same color as some edge of $T$. Let $R$ be the set of edges in $T$ that receive a label. The components of $T \setminus R$ create a partition $\pi$ of the vertices. We claim this partition contradicts the hypothesis. There are $n - 1 - \#(R)$ unlabeled edges in $T$, so $\#(\pi) - 1 = n - (n - \#(R)) - 1 = \#(R)$. Each edge $f$ that lies between two parts of $\pi$ also lies between the components of $T - e$ for some labeled edge $e$, hence by the minimality of $T$ edge $f$ has the same color as an edge in $R$. The number of different colors in $R$ is strictly less than $\#(R)$, since at least two of the edges are colored $c$ (i.e. labeled $a_1$). Thus, for this partition the number of colors between the parts is smaller than $\#(R) = \#(G) - 1$, contradicting the hypothesis.

There are many known polynomial algorithms for finding maximum weight common independent sets (see Schrijver [23], pages 705–707). Most notable are results by Edmonds [7] and Lawler [17]. Therefore there is a polynomial time algorithm for finding a rainbow spanning tree in $H_G$ and so a 2-trail traversal as well. The proof of Theorem 5.2 also gives a polynomial algorithm for finding a rainbow spanning tree of $H$ or finding a partition $\pi$ that demonstrates no rainbow spanning tree exists. 


6 Polynomial algorithm

Our results give a polynomial time algorithm that determines if a colored eulerian digraph \( G \) has a compatible circuit. If \( G \) does then the algorithm provides a compatible circuit and if not the algorithm provides a certificate that shows why such a compatible circuit does not exist.

Below is the outline of the algorithm.

**Input:** A colored eulerian digraph \( G \) with no nonfixable vertices of outdegree three with exactly three color classes.

**Output:** Compatible circuit of \( G \) or certificate which shows no such circuit exists.

**Step 1:** Check if \( \gamma(v) \leq \deg^+(v) \) for all vertices \( v \). If not then return as the certificate the vertex \( v \) such that \( \gamma(v) > \deg^+(v) \).

**Step 2:** Create the reduced colored eulerian digraph: split all vertices with \( \gamma(v) = \deg^+(v) \) according to Definition 2.1 and subdivide all loops to create graph \( G' \). Check whether \( G' \) is connected. If not then return a certificate indicating that \( G' \) is disconnected.

**Step 3:** Find the nonfixable vertices of \( G \) and construct the auxiliary graphs \( G_S \) and \( H_G \) from Definitions 4.1 and 4.2.

**Step 4:** Determine if \( H_G \) has a rainbow spanning tree. If not return as the certificate the partition \( \pi \) of the vertices of \( H_G \) that demonstrates the obstruction for the rainbow spanning tree. If \( H_G \) has a rainbow spanning tree find a rainbow spanning tree and call it \( R \).

**Step 5:** From \( R \) construct a 2-trail traversal \( E' \) with the two properties described in the proof of Theorem 4.5. Form \( G' \) (or \( \hat{G}' \)) by identifying vertices in \( G_S \) according to the 2-trail traversal of \( H_G \) and split appropriate vertices according to the proof of Theorem 4.5.

**Step 6:** Find an eulerian circuit \( T_0 \) in \( G' \) (where \( T_0 \) is not necessarily compatible). For each vertex \( v_i \in V(G') \), \( i = 1, \ldots, n \) perform the step below:

**Step 6a:** Construct the excursion graph \( L_{T_{i-1}}(v) \). Find a compatible circuit of \( L_{T_{i-1}}(v) \) and use the transitions to rearrange the excursions of \( T_{i-1} \) at \( v \) to form the eulerian circuit \( T_i \).

Return the compatible circuit \( T_n \) of \( G \).

Each of these steps can be completed in polynomial time. In particular, Step 4 follows from polynomial time algorithms for finding maximum common independent sets as discussed in the previous section. Step 6a can be computed in polynomial time by Proposition 3.9. Thus the entire algorithm runs in polynomial time.
7 Future work

Finally we provide some open questions about compatible circuits in eulerian digraphs.

**Question 1:** Theorem 4.5 provides necessary and sufficient conditions for the existence of compatible circuits when there are no nonfixable vertices of outdegree three. For these nonfixable vertices the component graph $H_G$ is not well defined, and the nonfixable vertices cause problems in the proof of Theorem 4.5 as discussed at the end of Section 4. Can we characterize the existence of a compatible circuit in a colored eulerian digraph $G$ with nonfixable vertices of outdegree three?

**Question 2:** The BEST Theorem [27, 28] provides a formula that counts the number of eulerian circuits in an eulerian digraph. Does there exist a formula to count the number of compatible circuits in a colored eulerian digraph?

**Question 3:** In a reduced eulerian digraph with no compatible circuit an eulerian circuit with a minimum number of monochromatic transitions must have a monochromatic transition at a nonfixable vertex. For a colored eulerian digraph (not necessarily reduced) with no compatible circuit what is the fewest number of monochromatic transitions in an eulerian circuit?

**Question 4:** For a digraph $G$ that is not eulerian the Chinese Postman Problem [8] is to find a closed walk in $G$ that travels each edge at least once and has the shortest length. Given an edge-colored strongly connected digraph (not necessarily eulerian) what is the minimum length of a closed walk with no monochromatic transitions?

The Chinese Postman Problem has many applications in routing problems. Introducing colors allows us to enforce additional restrictions on the routing. For instance we could color the edges of a road network such that a compatible circuit is a route for a mail truck that avoids left turns [18]. UPS [20] uses such routes to reduce the time of deliveries and number of accidents, saving millions of dollars.

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**References**


