Binary De Bruijn cycles under different equivalence relations

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Abstract

De Bruijn cycles are cyclic binary strings of length $n$ where all substrings of length $i$ are distinct. We present a generalization called $(\simeq, i)$-De Bruijn cycles that are defined for an equivalence relation $\simeq$ on substrings of length $i$. In this paper, binary $(\simeq, i)$-De Bruijn cycles under the equivalences of cyclic rotation, inverses, and flipping are examined. For the first two equivalences, we present exact solutions for all pairs $(i, n)$ for which $(\simeq, i)$-De Bruijn cycles of length $n$ exist; for the last equivalence a conjecture is made about which pairs hold.

1. Introduction

A De Bruijn $^1$ cycle is a cyclic binary string $S$ such that all substrings of $S$ of length $i$ are distinct. De Bruijn cycles have a wide variety of applications, such as Sanskrit memory wheels, pseudo-random number generation, error detection, and public-key cryptographic schemes [2]. One of the reasons for the usefulness of De Bruijn cycles is the large range of values $n$ where De Bruijn cycles of length $n$ exist. Flye-Sainte Marie, De Bruijn, and Good (see [5]) independently showed that for each $i$, a maximum cycle length of $2^i$ is possible. Yoeli, Bryant, Heath, Killick, Golomb, Welch, and Goldstein (see [4]) independently showed that a De Bruijn cycle of any cycle length less than the maximum of $2^i$ also exists, and Lempel [4] generalized this result to non-binary strings.

Chung et al. [2] have looked at more general relationships between substrings of non-binary strings, viewing them as permutations, partitions, or subsets of a finite set, and have had various degrees of success in determining which lengths of De Bruijn cycles are possible for a given substring length. Motivated by vertex colorings of...
a cycle, Cowen and Cheng [1] have looked at another variation of De Bruijn cycles where substrings are equivalent if and only if they are equal or flips of one another.

In this paper, we present a generalization of De Bruijn cycles, called \((\simeq, i)\)-De Bruijn cycles, defined for an equivalence relation \(\simeq\) on substrings of length \(i\). Normal De Bruijn cycles are thus \((=, i)\)-De Bruijn cycles, where the relation \(=\) is straight equality between substrings. Using our generalization with three other equivalence relations, we attempt to determine for which lengths \(n\) \((\simeq, i)\)-De Bruijn cycles exist.

Let \(A = \{0, 1, \ldots, a - 1\}\) be an alphabet of \(a\) elements from which cyclic strings \(S = (s_0, s_1, \ldots, s_{n-1})\) are created. All subscripts are viewed modulo \(n\), and two strings \(S\) and \(T = (t_0, t_1, \ldots, t_{n-1})\) of length \(n\) are considered the same if there exists an integer \(k\) such that \(s_j = t_{j+k}\) for all \(0 \leq j < n\). Consider all substrings \(S_j = (s_j, s_{j+1}, \ldots, s_{j+i-1})\) of length \(i\), \(0 \leq j < n\), of a string \(S\) of length \(n\) (assume that \(i \leq n\)), and define an equivalence relation \(\simeq\) between two such substrings. A cyclic string \(S\) is called a \((\simeq, i)\)-De Bruijn cycle if \(S_j \simeq S_k\) implies that \(j \equiv k\) (mod \(n\)). The motivating problem is to find, given \(A\) and \(\simeq\), all pairs \((i, n)\); \(1 \leq i \leq n\), such that a \((\simeq, i)\)-De Bruijn cycle of length \(n\) exists.

In this work, we will be considering only binary cyclic strings. Thus, we assume \(A = \{0, 1\}\). The notation \(\overline{b}\) will be used to denote the inverse of a bit or binary string. In this paper, we consider the three equivalence relations of cyclic rotation \(\sim\), inversion \(\simeq\), and flipping \(\approx\) on binary cyclic strings. We will show that \((\sim, i)\)-De Bruijn cycles of length \(2i\) exist only for \(i = 1\) and \(i \geq 7\), and that \((\simeq, i)\)-De Bruijn cycles of length \(i \leq n \leq 2^{i-1}\), \(n \neq 2^{i-1} - 2\), exist for \(i \geq 4\). For flipping, we conjecture that a range of values will be possible, followed by a gap, and then a single maximum value.

2. Cyclic rotation

The fact that two \((\simeq, i)\)-De Bruijn cycles are considered equivalent if they are cyclic rotations of one another suggests examining cyclic rotations as an equivalence relation for substrings. Thus, in this section we will always use the equivalence relation \(\sim\) for substrings.

**Definition 1.** Two strings \(T\) and \(U\) of length \(i\) are cyclic rotations of one another \((T \sim U)\) if there exists an integer \(r\) such that \(t_l = u_{l+r}\) for all \(0 \leq l < i\).

**Lemma 1.** If a \((\sim, i)\)-De Bruijn cycle of length \(n\) exists, then \(n = 2i\).

**Proof.** Let \(S\) be a \((\sim, i)\)-De Bruijn cycle of length \(n\). First note that \(s_{j+i} = \overline{s_i}\). Otherwise, \(S_{j+1}\) would be a cyclic rotation of \(S_j\). Hence, \(S_j = \overline{S_{j+i}} = S_{j+2i}\). Since \(S\) is a \((\sim, i)\)-De Bruijn cycle, this implies \(j \equiv j + 2i (\text{mod } n)\), and thus \(n\) divides \(2i\). If \(n = i\), then every substring is the entire string and is equivalent under cyclic rotation. Along with the condition that \(n \geq i\), this shows that \(n = 2i\). \(\Box\)
Definition 2 (Lempel [3]). A cyclic binary string $S$ of length $2i$ is called a self-dual cycle if and only if $s_j = \overline{s_{j+i}}$ for all $0 \leq j < 2i$.

To show that there exist self-dual $(\sim,i)$-De Bruijn cycles of length $2i$ for large enough $i$, an explicit construction will be given. First, some lemmas are needed.

Lemma 2. In a self-dual cycle of length $2i$, let $S_j$ and $S_{j+l}$, $0 < l \leq i$, overlap. If $S_j \simeq S_{j+l}$, then the overhang substrings $H_j = (s_j, s_{j+1}, \ldots, s_{j+l-1})$ and $H_{j+i} = (s_{j+i}, s_{j+i+1}, \ldots, s_{j+i+l-1})$ each have equal numbers of 0s and 1s, and this number is the same for both $H_j$ and $H_{j+i}$.

Proof. Let $V = (s_{j+l}, s_{j+i+1}, \ldots, s_{j+i+l-1})$ be the overlap substring of $S_j$ and $S_{j+l}$. Note that if $l = i$, then $V$ is an empty string.

Since $S_{j+i} = \overline{S_j}$, we have $H_{j+i} = \overline{H_j}$. Let $a$ denote the number of 0s in $H_j$ and $b$ the number of 1s. Let $c$ denote the number of 0s in the overlap substring $V$. Then $a + c = c + b$ since the number of 0s in $S_j = H_j V$ must equal the number of 0s in $S_{j+i} = VH_{j+i}$. Therefore, $a = b$. □

Lemma 3. Let $S$ be a self-dual cycle of length $2i$, $H_j = (s_j, s_{j+1}, \ldots, s_{j+l-1})$ be a substring of length $l$, $1 \leq l \leq i$, with equal numbers of 0s and 1s, and $V_j = (s_{j+l}, s_{j+i+1}, \ldots, s_{j+i+l-1})$ be a substring of length $i - l$. Then $H_j V_j \sim V_j H_{j+i}$ if and only if $H_{j+i} V_{j+l+i} \sim V_{j+l+i} H_j$.

Proof. Since $S$ is a self-dual cycle, then $H_{j+i} = \overline{H_j}$ and $V_{j+l+i} = \overline{V_j}$ (again, if $l = i$, then $V_j$ is an empty string). Thus, $H_j V_{j+l} = \overline{H_{j+i} V_{j+l+i}}$ and $V_{j+i} H_{j+i} = \overline{V_{j+l+i} H_j}$. Using the fact that $T \sim U$ if $\bar{T} \sim \bar{U}$, we have $H_j V_{j+l} \sim V_{j+l+i} H_j$ if and only if $H_{j+i} V_{j+l+i} \sim V_{j+l+i} H_j$. □

The usefulness of this lemma comes in proving that a self-dual cycle $S$ is a $(\sim,i)$-De Bruijn cycle. Only substrings $H_j$ starting in $(s_0, s_1, \ldots, s_{i-1})$ with equal number of 0s and 1s now need to be examined to see if $H_j V_{j+l}$ will lead to an equivalence.

Theorem 1. There exists a $(\sim,i)$-De Bruijn cycle of length $2i$ for $i \geq 7$ and $i = 1$. No $(\sim,i)$-De Bruijn cycle exists for $2 \leq i < 7$. 


Table 1

<table>
<thead>
<tr>
<th>$H_j$</th>
<th>$S_j$</th>
<th>$S_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{i-4} = 01$</td>
<td>$S_{i-4}$</td>
<td>$S_{i-2}*$</td>
</tr>
<tr>
<td>$H_{i-3} = 10$</td>
<td>$S_{i-3}*$</td>
<td>$S_{i-1}$</td>
</tr>
<tr>
<td>$H_{i-1} = 01$</td>
<td>$S_{i-1}*$</td>
<td>$S_{i+1}$</td>
</tr>
<tr>
<td>$H_{i-2} = 0011$</td>
<td>$S_{i-2}*$</td>
<td>$S_{i+2}$</td>
</tr>
<tr>
<td>$H_{i-3-i} = 001001101101$</td>
<td>$S_{i-3-i}*$</td>
<td>$S_{i+1+i}$</td>
</tr>
</tbody>
</table>

Proof. For a given $i \geq 7$, construct $S$ as follows:

$$S = 00 \ldots 0 10011 \ldots 1 0111.$$

Table 1 shows all substrings $H_j$ starting in $S_0$ with equal numbers of 0s and 1s. In the last case, $H_{i-3-i}$ is a substring of length $2l + 4$ where $0 \leq l \leq i/2 - 2$. These substrings are possible overhang substrings, and, by Lemma 3, only the two overlapping substrings $S_j = H_l V_{i-l}$ and $S_k = V_{i-l} H_{i-l}$ need to be checked for equivalence. In each case, one of $S_j$ and $S_k$ has a block of $i - 3$ 1s (denoted with an asterisk in the table), and the other does not, so $S_j \sim S_k$. Thus, no two distinct substrings of $S$ are equivalent.

An exhaustive computer search performed on $2 \leq i < 7$ found no $(\sim, i)$-De Bruijn cycles. For $i = 1$, the string 01 is the only $(\sim, i)$-De Bruijn cycle.

It is interesting to note that for $3 \leq i \leq 5$, the construction breaks down because the $i - 3$ block of 1s is not large enough to be distinct. For $i = 6$, all of the above cases hold, but another overhang string exists at $S_4 = 001110$ which leads to an equivalence.

3. Inversion

We now consider the common operation of binary inversion as an equivalence relation on substrings.

Definition 3. For strings $T$ and $U$, $T \simeq U$ if $T = U$ or $T = \overline{U}$.

Our approach is to find circuits in the standard De Bruijn graphs that correspond to $(\sim, i)$-De Bruijn cycles. We will then use a homomorphism defined by Lempel to map these circuits into other circuits that are easily characterizable.

Definition 4 (van Lint and Wilson [5]). The De Bruijn graph $G_i$ of order $i$ is a digraph with vertex set $V = \{0, 1\}^{i-1}$, the set of all binary strings of length $i - 1$. An arc goes from a vertex $X = (x_0, x_1, \ldots, x_{i-2})$ to a vertex $Y = (y_0, y_1, \ldots, y_{i-2})$ if and only if $x_{j+1} = y_j$ for all $0 \leq j \leq i - 2$. 

In other words, an arc exists from \( X \) to \( Y \) if the last \( i - 2 \) bits of \( X \) are the same as the first \( i - 2 \) bits of \( Y \).

An arc from \( X \) to \( Y \) in \( G_i \) can be labeled as a substring \((x_0, x_1, \ldots, x_{i-2}, y_{i-2})\). A \((\simeq, i)\)-De Bruijn cycle \( S \) then corresponds to a circuit \( C \) in the De Bruijn graph \( G_i \), where each substring corresponds to an arc. Each bit \( s \) of \( S \) corresponds to an arc \( A \) of \( C \), where the first bit of \( A \) is \( s \).

The following is a useful transformation of a De Bruijn graph.

**Definition 5** (van Lint and Wilson [5]). The *line digraph* (or *double*) \( G_i^* \) of a De Bruijn graph \( G_i \) is a digraph where the arc set of \( G_i \) is the vertex set of \( G_i^* \). An arc exists from a vertex \( X \) to a vertex \( Y \) in \( G_i^* \) if and only if the terminal vertex of the edge corresponding to \( X \) in \( G_i \) is the initial vertex of the edge corresponding to \( Y \) in \( G_i \).

Clearly, \( G_i^* \cong G_{i+1} \). This observation and a standard fact about line digraphs forms the following lemma.

**Lemma 4.** A circuit \( C \) in \( G_i \) becomes a cycle of the same length in \( G_i^* \cong G_{i+1} \) under the line digraph transformation.

Lempel [3] defined a homomorphism from \( G_i \) to \( G_{i-1} \) where inverse arcs and vertices in \( G_i \) that are equivalent under \( \simeq \) are mapped to the same arc or vertex, respectively, in \( G_{i-1} \). We will use this homomorphism to characterize circuits in \( G_i \) corresponding to \((\simeq, i)\)-De Bruijn cycles.

**Theorem 2** (Lempel [3]). Define \( D: G_i \to G_{i-1} \) for \( i \geq 2 \) as follows. For a vertex \( X = (x_0, x_1, \ldots, x_{i-1}) \) in \( G_i \), \( D(X) \) is a vertex \( Z = (z_0, z_1, \ldots, z_{i-2}) \) in \( G_{i-1} \) where

\[
z_j = x_j \oplus x_{j+1}, \quad 0 \leq j \leq i - 2
\]

and \( \oplus \) is addition mod 2. Then \( D \) is a homomorphism from \( G_i \) to \( G_{i-1} \), and, for two vertices (arcs) \( X \) and \( Y \) in \( G_i \), \( D(X) = D(Y) \) if and only if \( X \simeq Y \).

It is interesting to note that the homomorphism generates the Gray or reflected binary code [6] for binary strings of length \( i - 2 \).

Lempel’s homomorphism can be used to find circuits in \( G_i \) that correspond to \((\simeq, i)\)-De Bruijn cycles using the following definition and characterization.

**Definition 6** (Lempel [3]). Let \( E \) be a subgraph of \( G_i \). Then the *weight* \( W(E) \) of \( E \) is the number of arcs \( A = (a_0, a_1, \ldots, a_{i-1}) \) in \( E \) such that \( a_0 = 1 \).

If the subgraph \( E \) is a circuit \( C \), then \( W(C) \) is also the number of 1s in the \((\simeq, i)\)-De Bruijn cycle \( S \) that corresponds to \( C \).
Theorem 3 (Lempel [3]). Let $C'$ be a circuit of length $n$ in $G_{i-1}$. Then $C'$ is the homomorphic image under $D$ of a circuit $C$ in $G_i$ of length $n$ such that no two distinct arcs of $C$ are inverses of one another if and only if $W(C')$ is even.

Note that if $C$ is a circuit of length $n$ in $G_i$ without inverses, then so is $\tilde{C}$, and $D(\tilde{C}) = D(C)$.

The problem of finding $(z, i)$-De Bruijn cycles of length $n$ is thus reduced to finding a standard De Bruijn cycle $C'$ of length $n$ in $G_{i-1}$ that has even weight.

Theorem 4 (Flye-Sainte Marie, De Bruijn, and Good, see van Lint and Wilson [5]). There exists a circuit in $G_i$ of length $2^i$.

Theorem 5 (see Lempel [4]). For $1 \leq n \leq 2^{i-1}$, there exists a circuit $C'$ of length $n$ in $G_{i-1}$ of even weight.

Lemma 5. For all odd $n$, $1 \leq n < 2^{i-1}$, there exists a circuit $C'$ of length $n$ in $G_{i-1}$ of even weight.

Proof. Given $n$ odd with $1 \leq n < 2^{i-1}$, Theorem 5 implies that there exists a circuit $C'$ in $G_{i-1}$ of length $n$. If $W(C')$ is even, then the conclusion holds. If $W(C')$ is odd, then $C'$ contains an even number of 0s, and hence $\overline{C'}$ contains an even number of 1s. Thus, $\overline{C'}$ satisfies the desired conditions. □

To prove that circuits of even length and of even weight exist in $G_{i-1}$, a few more lemmas are needed.

Lemma 6. The removal of vertices $X = 101\ldots$ and $\tilde{X} = 010\ldots$ from $G_i$, $i > 3$, does not disconnect $G_i$.

Proof. To prove that $G_i - \{X, \tilde{X}\}$ is connected, it is sufficient to prove that, if there exists an arc from a vertex $Y$ to $X$ or $\tilde{X}$ and an arc from $X$ or $\tilde{X}$ to $Z$, then there exists a directed path from $Y$ to $Z$ without passing through $X$ or $\tilde{X}$.

Suppose that $i$ is odd. Then $X = 101\ldots01$, and $\tilde{X} = 010\ldots10$. Let $Y_1 = 1101\ldots010$, $Y_2 = Y_1 = 0010\ldots101$, $Z_1 = 101\ldots0100$, and $Z_2 = Z_1 = 010\ldots101$. Arcs exist from $Y_1$ to $X$, from $Y_2$ to $\tilde{X}$, from $\tilde{X}$ to $Z_1$, from $X$ to $Z_2$, from $X$ to $Y$, and from $\tilde{X}$ to $X$. Then there exists an arc $11010\ldots1011$ connecting $Y_1$ directly to $Z_1$. A directed path in $G_i$ corresponds to a (non-cyclic) binary string, with arcs corresponding to substrings. We can construct a string $T$ representing a directed path $P$ from $Z_1$ to vertex $00101\ldots1$ by adding $i - 3$ alternating 1010 bits to $Z_1$, and then by adding a 1, $P$ reaches $Z_2$. Since the vertices in $P$ always have two consecutive 0s except for $Z_2$, $P$ never passes through $X$ or $\tilde{X}$. Using the same argument except with inverses shows that directed paths exist from $Y_2 = Y_1$ to $Z_1$ and $Z_2$.

If $i$ is even, then a similar argument using the same pattern holds. □
Lemma 7. For all \( i > 2 \), there exists a circuit in \( G_i \) of length \( 2^i \) such that the arcs 0101\ldots and 1010\ldots are not adjacent.

**Proof.** By Theorem 4, there exists a circuit \( C \) in \( G_i \) of length \( 2^i \). If the arcs \( A_1 = 0101\ldots \) and \( A_2 = 1010\ldots \) are not adjacent, then the conclusion follows immediately. Suppose, without loss of generality, that \( A_2 \) is the immediate successor of \( A_1 \) in \( C \). Let \( X = 101\ldots \) be the terminal vertex of \( A_1 \) and the initial vertex of \( A_2 \), and let \( B_1 \) denote the other incoming arc of \( X \) and \( B_2 \) the other outgoing arc. As a sequence of arcs, \( C \) can be written as \( (\ldots, A_1, A_2, \ldots, B_1, B_2, \ldots) \). \( C \) can be divided into two walks \( P_1 \) and \( P_2 \), where \( P_1 \) is defined by the arcs going from \( A_2 \) to \( B_1 \) in \( C \) and \( P_2 \) is defined by the arcs going from \( B_2 \) to \( A_1 \). Let \( M_1 \) be the set of vertices through which \( P_1 \) passes, excluding \( X \) and \( \bar{X} \), and let \( M_2 \) be the set of vertices through which \( P_2 \) passes, again excluding \( X \) and \( \bar{X} \). Since the removal of \( X \) and \( \bar{X} \) from \( G_i \) does not disconnect \( G_i \) for \( i > 2 \) (by Lemma 6), \( M_1 \cap M_2 \) cannot be empty. Let \( Y \in M_1 \cap M_2 \), with incoming arcs \( A'_1 \) in \( P_1 \) and \( B'_1 \) in \( P_2 \) and outgoing arcs \( B'_2 \) in \( P_1 \) and \( A'_1 \) in \( P_2 \).

Construct a new circuit \( C' \) from \( C \) by switching successive arcs at \( X \) and \( Y \), giving \( C' = (\ldots, A_1, B_2, A_2, B_1, \ldots, B_1, A_2, A_1, \ldots) \). Since \( \bar{X} \) was excluded from \( M_1 \) and \( M_2 \), arcs \( A_1 \) and \( A_2 \) are not adjacent in \( C' \).

The following two lemmas without any statements about weight were used by Lempel in [3] to prove Theorem 5 for non-binary strings. Lempel’s proof for Lemma 9 is given here, but with extra observations about the weights of \( E \) and \( C \).

Lemma 8 (Lempel [4]). If \( W, X, Y, \) and \( Z \) are vertices in \( G_i \) and arcs exist from \( W \) to \( X \), from \( W \) to \( Y \), and from \( Z \) to \( X \), then there exists an arc from \( Z \) to \( Y \).

Lemma 9 (Lempel [4]). Let \( E \) be a subgraph of \( G_i \) with \( k \) arcs (not necessarily connected) where every vertex \( X \) in \( E \) has in-degree \( d_i(X) \) and out-degree \( d_o(X) \) of 1. Then \( G_i \) contains a circuit \( C \) of length \( 2^i - k \) where \( W(C) \) has the same parity as \( W(E) \).

**Proof.** Let \( E_1 = E \), and let \( C_1 \) be the complement of \( E_1 \) with respect to \( G_i \). Obviously, \( C_1 \) has \( 2^i - k \) arcs and \( d_i(X) = d_o(X) \geq 1 \) in \( C_1 \) for all \( X \) in \( G_i \). Since \( W(C_1) = 2^{i-1} - W(E_1) \), \( C_1 \) also has the same parity as \( W(E_1) \). Let \( C^i \) (\( i = 1, 2, \ldots, p \geq 1 \)) be the maximal connected components of \( C_1 \). If \( p = 1 \), then the conclusion holds immediately. Suppose that \( p > 1 \). Then \( E_1 \) contains an arc whose endpoints belong to distinct components of \( C_1 \). Let \( \langle W, X \rangle \) be such an arc from \( W \in C^1 \) to \( X \in C^2 \). Now, \( C^1 \) contains at least one arc \( \langle W, Y \rangle \), and \( C^2 \) contains at least one arc \( \langle Z, X \rangle \). Using Lemma 8, there exists an arc \( \langle Z, Y \rangle \in E_1 \) which joins \( C^1 \) with \( C^2 \). Construct \( C_2 \) from \( C_1 \) by replacing \( \langle W, X \rangle \) and \( \langle Z, Y \rangle \) with \( \langle W, Y \rangle \) and \( \langle Z, X \rangle \), and construct \( C_2 \) from \( C_1 \) by replacing \( \langle W, Y \rangle \) and \( \langle Z, X \rangle \) with \( \langle W, X \rangle \) and \( \langle Z, Y \rangle \). Since the two outgoing arcs of any vertex have the same initial bit, switching the two arcs between two subgraphs does not change the weight of either subgraph. Thus, \( W(C_2) = 2^{i-1} - W(E) \).
Table 2

<table>
<thead>
<tr>
<th>n</th>
<th>cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>none</td>
</tr>
<tr>
<td>4</td>
<td>0011</td>
</tr>
<tr>
<td>6</td>
<td>none</td>
</tr>
<tr>
<td>8</td>
<td>00011101</td>
</tr>
</tbody>
</table>

Clearly, for every vertex $X$ in $C_2$, $d_i(X) = d_o(X) \geq 1$, and $C_2$ has only $p - 1$ maximal connected components. After applying the same procedure $p - 2$ times more, one obtains a connected $C_p$ with $2^i - k$ arcs where $d_i(X) = d_o(X)$ for all $X \in C_p$, and $W(C_p) = 2^{i-1} - W(E)$. □

Lemma 10. For all even $n$ and $i \geq 3$ where $4 \leq n \leq 2^i$ and $n \neq 2^i - 2$, there exists a circuit $C$ of length $n$ in $G_i$ that has even weight.

Proof. The proof is by induction on $i$.

Basis step: Circuits of even length $n$ and even weight for $i = 3$, are shown in Table 2.

Induction step: Assume that an even-weight circuit of length $n$ exists in $G_{i-1}$, $i > 3$, for all even $n$ where $4 \leq n \leq 2^i - 1$, $n \neq 2$, and $n \neq 2^i - 2$.

Let $n = 2^i - k$, where $k$ is even.

If $k \geq 2^i - 1$ and $n \neq 2^i - 2$, then, by hypothesis, there exists a circuit $C'$ in $G_i$ of length $2^i - k$ that has even weight. Using the line digraph transformation, $C'$ is a circuit of even weight in $G_i$.

If $k = 2^i - 1 + 2$, then there exists a circuit $C'$ in $G_{i-1}$ of length $2^i - 1$. Removing the 00...0 loop and the 0101... and 1010... arcs from $C'$ and using the line digraph transformation forms a circuit $C''$ in $G_i$ of length $2^i - 1 - 3$ and odd weight. Adding the 11...1 loop to $C''$ creates a circuit $C$ in $G_i$ that has length $2^i - 2$ and even weight.

If $k = 2^i - 1 - 2$, then by Lemma 7, there exists a circuit $C'$ in $G_{i-1}$ of length $2^i - 1$ where the $A_1 = 0101...$ and $A_2 = 1010...$ arcs are not adjacent. Removing the 00...0 loop from $C'$ and using the line digraph transformation forms a circuit $C''$ in $G_i$ of length $2^i - 1 - 1$ and even weight. Since $A_1$ and $A_2$ were not adjacent in $C'$, the arcs $B_1 = 0101...$ and $B_2 = 1010...$ in $G_i$ are not in $C''$. Adding the 11...1 loop and arcs $B_1$ and $B_2$ to $C''$ forms a circuit $C$ in $G_i$ that has length $2^i - 2 + 2$ and even weight.

If $k = 0$, then an Eulerian circuit of $G_i$ has length $n = 2^i$ and has weight $2^i - 1$.

Suppose that $4 \leq k \leq 2^i - 4$. By hypothesis, there exists a circuit $C'$ in $G_{i-1}$ of length $2^i - 1 - k$ that has even weight. The complement $C''$ of $C'$ with respect to $G_{i-1}$ is either a circuit or a collection of circuits, and has $k$ arcs and even weight. Using the line digraph transformation, $C''$ becomes a cycle or a collection of cycles in $G_i$ with $k$ arcs and even weight. Using Lemma 9, there exists a circuit $C$ in $G_i$ of length $2^i - k$ which also has even weight. □
Lemma 11. For \( i > 1 \), no circuit of length 2 or \( 2^i - 2 \) in \( G_i \) has even weight.

Proof. The only circuit of length 2 in \( G_i \) is the circuit of 0101... and 1010..., which has odd weight. The only ways to obtain a circuit of length \( 2^i - 2 \) in \( G_i \) are to remove the 00...0 and 11...1 loops or to remove the circuit of 0101... and 1010.... Both of these removals result in circuits of odd weight.  

Theorem 6. For all \( 1 \leq n \leq 2^i - 1 \), where \( n \neq 2^{i-1} - 2 \), \( n \neq 2 \), and \( i \geq 4 \), there exists a circuit \( C \) of length \( n \) in \( G_i \) such that no two distinct arcs are inverses of one another. Thus, \((\approx, i)\)-De Bruijn cycles exist for these lengths when \( i \leq n \). No \((\approx, i)\)-De Bruijn cycles exist of length \( 2^{i-1} - 2 \) or when \( i = 2, 3 \). The strings 0 and 1 are the only \((\approx, 1)\)-De Bruijn cycles.

Proof. The first part follows from applying Lemma 3 to the circuits of even weight obtained from Lemmas 5 and 10. The second part follows from applying Lemma 3 to Lemma 11, and the extra cases are obtained by exhaustion.  

4. Flipping

A question posed by Cowen and Cheng [1] on the vertex coloring of cycles has motivated consideration of another equivalence relation. Paths along the cycle are analogous to substrings of a De Bruijn cycle, and the natural equivalence relation between two paths is isomorphism. Cowen and Cheng thus consider the following relation.

Definition 7. \( S_j \approx S_k \) if \( s_{j+l} = s_{k+l} \) or \( s_{j+l} = s_{k+(i-l-1)} \) for all \( 0 \leq l < i \).

Conjecture 5.1 of [1] states (in our terminology) that, for fixed \( i > 0 \) and using the relation above, the minimum cardinality of the alphabet \( A \) for a pair \((i,n)\) to have a \((\approx, i)\)-De Bruijn cycle of length \( n \) is monotonically non-decreasing as \( n \) increases. However, our experimental data obtained by exhaustive computer search for \( 4 \leq i \leq 7 \) show the lengths for which binary \((\approx, i)\)-De Bruijn cycles exist. No cycles exist for lengths not listed in Table 3.

These data suggest that a gap exists in the lengths for which binary \((\approx, i)\)-De Bruijn cycles exist for \( i > 4 \). We thus make the following conjecture based on this observation.

Table 3

<table>
<thead>
<tr>
<th>( i )</th>
<th>lengths where cycles exist</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>6...10, 12</td>
</tr>
<tr>
<td>6</td>
<td>6...24, 30</td>
</tr>
<tr>
<td>7</td>
<td>7...56, 60</td>
</tr>
</tbody>
</table>
Conjecture 1. Let \( A = \{0, 1\} \). Then for all \( i > 4 \), there exists a range of consecutive integers starting at \( i \) for which a binary \((\approx, i)\)-De Bruijn cycle exists, then a gap for which no such cycle exists, and then a single maximum value where a cycle exists.

If the gap in the range of values exists, then the minimum cardinality of \( A \) for a \((\approx, i)\)-De Bruijn cycle when the cycle length \( n \) is maximum is less than the minimum cardinality of \( A \) for \( n - 1 \). Thus, if true, Conjecture 1 would refute Conjecture 5.1 of [1]. However, attempts of the author to prove either conjecture have not succeeded.

5. Conclusion

The formulation of \((\approx, i)\)-De Bruijn cycles provides a source for many new questions about cyclic strings. The results and conjectures of this paper can be extended to non-binary cyclic strings, and \((\approx, i)\)-De Bruijn cycles under other equivalence relations can also be examined. A further generalization of De Bruijn cycles might be to impose a metric on substrings, such as Hamming distance.

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References