RAMSEY-TYPE NUMBERS FOR DEGREE SEQUENCES

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Abstract. A (finite) sequence of non-negative integers is graphic if it is the
degree sequence of some simple graph $G$. In this paper, we introduce a Ramsey-
type parameter for degree sequences. Given graphs $G_1$ and $G_2$, we define the
potential-Ramsey number $r_{pot}(G_1, G_2)$, as the smallest integer $n$ such that for
every $n$-term graphic sequence $\pi$, there is some graph $G$ with degree sequence $\pi$
with $G_1 \subseteq G$ or with $G_2 \subseteq G$. Bounded above by the well-studied classical
Ramsey number, we consider situations where equality holds, and give exact
values for $r_{pot}(K_n, K_t), r_{pot}(C_n, K_t), r_{pot}(P_n, K_t)$.

1. Introduction and Preliminaries

If $u$ and $v$ are adjacent vertices in $G$, we will write $u \sim v$ and say also that
$uv$ is an edge of $G$. For a subgraph $H$ and a vertex $v$ in $G$, $N_H(v)$ denotes those
neighbors of $v$ lying in $H$ and we let $d_H(v) = |N_H(v)|$. Similarly, given vertices $u$
and $v$ in $H$, we let $dist_H(u, v)$ denote the distance from $u$ to $v$ in $H$.

A nonegative integer sequence $(d_1, \ldots, d_n)$ is graphic if it is the degree sequence
of some (simple) graph $G$. In that case, we say that $G$ realizes $\pi$ or is a realization
of $\pi$, and we write $\pi = \pi(G)$ or $G = G(\pi)$. A graphic sequence $\pi$ is unigraphic
if all realizations of $\pi$ are isomorphic, and we denote $\pi\overrightarrow{=} = (d_1, \ldots, d_n) = (n - 1 -
d_1, \ldots, n - 1 - d_n)$ and when convenient we will reorder the terms in nonincreasing
order. For a fixed graph $H$, a graphic sequence $\pi$ is potentially $H$-graphic if there
is some realization of $\pi$ that contains $H$ as a subgraph.

Given graphs $G_1$ and $G_2$, the classical Ramsey number $r(G_1, G_2)$ is the minimum
integer $n$ such that for any graph $G$ of order $n$, either $G_1 \subseteq G$ or $G_2 \subseteq G$.
Exact values of $r(G_1, G_2)$ are known for very few collections of graphs - for a thorough
dynamic survey see [15]. As the classical Ramsey problem has proved to be difficult,
a number of variants and relaxations, many of which are still quite challenging, have
arisen and been the subject of a great deal of study (for a small sample of recent
results, see [1, 3, 8, 10, 11]).
In this paper, we introduce a Ramsey-type parameter for degree sequences. Given a graphic sequence \( \pi \), we will write \( \pi \rightarrow (G_1, G_2) \) if either \( \pi \) is potentially \( G_1 \)-graphic or \( \pi \) is potentially \( G_2 \)-graphic. The potential-Ramsey number, \( r_{\text{pot}}(G_1, G_2) \), is the smallest integer \( n \) such that for every \( n \)-term graphic sequence \( \pi \), \( \pi \rightarrow (G_1, G_2) \). Alternatively, \( r_{\text{pot}}(G_1, G_2) \), is the smallest integer \( n \) such that for every \( n \)-term graphic sequence \( \pi \), there is some graph \( G = G(\pi) \) with \( G_1 \subseteq G \) or with \( G_2 \subseteq \overline{G} \). In the sections that follow, we will give several bounds on \( r_{\text{pot}}(G_1, G_2) \) and also determine exact values for a number of families of graphs.

2. Bounds

The first claim in this section establishes a concrete link between the classical Ramsey number \( r \) and the Ramsey-potential number \( r_{\text{pot}} \).

**Claim 1.** For any graphs \( G_1 \) and \( G_2 \),
\[
 r_{\text{pot}}(G_1, G_2) \leq r(G_1, G_2).
\]

As a number of our results will subsequently demonstrate, this bound is often far from optimal. However, there are a number of instances for which the bound is sharp. For instance, in light of [9], we have
\[
 r_{\text{pot}}(K_1, n, K_1, t) = r(K_1, n, K_1, t) = \begin{cases} 
 n + t - 1 & n, t \text{ both even}, \\
 n + t & \text{otherwise}.
\end{cases}
\]

Furthermore, the following lemma demonstrates that in certain cases it is straightforward to determine when equality holds in Claim 1.

**Lemma 2.1.** Let \( r = r(G_1, G_2) \) and let \( G \) be a graph of order \( r - 1 \) such that \( G_1 \not\subseteq G \) and \( G_2 \not\subseteq \overline{G} \). If \( \pi(G) \) is unigraphic, then \( r(G_1, G_2) = r_{\text{pot}}(G_1, G_2) \).

This allows us to obtain several interesting results. For instance, it is well known that \( r(K_3, K_3) = 6 \) and that \( C_5 \) is the unique graph on five vertices containing no triangle and no independent set of size three. Since \( \pi(C_5) = (2, 2, 2, 2, 2) \) is unigraphic, we conclude that \( r_{\text{pot}}(K_3, K_3) = r(K_3, K_3) = 6 \).

In [7], it was shown that for \( n \geq m \geq 2 \), \( r(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1 \), where the lower bound was established via consideration of the graph \( G = K_{n-1} \cup K_{\left\lfloor \frac{m}{2} \right\rfloor} \). As \( \pi(G) \) is unigraphic, we obtain that
\[
 r_{\text{pot}}(P_n, P_m) = r(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.
\]

Anything else where equality holds? Check dynamic survey and literature.

Next we give a bound on \( r_{\text{pot}}(G, K_n) \) for a number of choices of \( G \). The 1-dependence number of a graph \( G \) [6], denoted \( \alpha^{(1)}(G) \), is the maximum order of an induced subgraph \( H \) of \( G \) with \( \Delta(H) \leq 1 \).

**Theorem 2.2.** Let \( G \) be a graph of order \( n \) with \( \alpha^{(1)}(G) \leq n - 1 \) and \( t \geq 2 \). Then,
\[
 r_{\text{pot}}((G, K_t)) \geq 2t + n - \alpha^{(1)}(G) - 2.
\]
Proof. Let $\ell = n - \alpha^{(1)}(G) - 1$ and consider $\pi = \pi(K_\ell \vee (t - 1)K_2)$, which is unigraphic. The result follows from two simple observations. First, $\pi$ is uniquely realized by $(K_{2n-2} - (t - 1)K_2) \cup K_2$ which contains no $K_t$. Secondly, any copy of $G$ lying in the unique realization of $\pi$ requires at least $\alpha^{(1)}(G) + 1$ vertices from the $t - 1$ independent edges, which is impossible as any such collection of vertices would necessarily induce a subgraph of $G$ with order at least $\alpha^{(1)}(G) + 1$ and maximum degree at most one. \hfill $\square$

This theorem yields a straightforward corollary, springing from the fact that for any graph $G$ of order $n$, $\pi((K_{n-1} \cup \overline{K}_{t-2}) \not\rightarrow (G, K_t)$.

**Corollary 2.3.** Let $G$ be a graph of order $n$ with $\alpha^{(1)}(G) \leq n - 1$ and let $t \geq 2$. Then,

$$r_{pot}((G, K_t)) \geq \max\{2t + n - \alpha^{(1)}(G) - 2, n + t - 2\}.$$  

As we will see in the sections that follow, this bound is accurate for $G = K_n, C_n$ and $P_n$.

3. $r_{pot}(K_n, K_t)$

Few exact values of $r(K_n, K_t)$ are known, and it is one of the foremost problems in combinatorics to determine even $r(K_5, K_5)$. Despite this, in this section, we determine $r_{pot}(K_n, K_t)$ for $n \geq t \geq 2$. The following results will be useful as we proceed.

**Theorem 3.1** (R. Luo [12]). Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $d_3 \geq 2$ and $d_n \geq 1$. Then $\pi$ is potentially $C_3$-graphic if and only if $\pi \neq (2, 2, 2)$ and $\pi \neq (2, 2, 2, 2)$.

**Theorem 3.2** (J-H Yin and J-S Li [16]). Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence and $k \geq 1$ be an integer.

(a) If $d_k \geq k - 1$ and $d_{2k} \geq k - 2$ then $\pi$ is potentially $K_k$-graphic.

(b) If $d_k \geq k - 1$ and $d_i \geq 2(k - 1) - i$ for $1 \leq i \leq k - 1$, then $\pi$ is potentially $K_k$-graphic.

We are now ready to prove the main result of this section.

**Theorem 3.3.** For $n \geq t \geq 3$,

$$r_{pot}(K_n, K_t) = 2n + t - 4$$

except when $n = t = 3$, in which case $r_{pot}(K_3, K_3) = 6$.

Proof. The case where $n = t = 3$ has already been discussed in Section 2, so we may assume that $n \geq 4$ and $t \geq 3$. The fact that $r_{pot}(K_n, K_t) \geq 2n + t - 4$ follows from Theorem 2.2 and/or consideration of $\pi((K_{2n-2} - (n - 1)K_2) \cup \overline{K}_{t-3})$, which is unigraphic.
Therefore, let \( \pi = (d_1, \ldots, d_k) \) be a nonincreasing graphic sequence of length \( k = 2n + t - 4 \) with complementary sequence \( \overline{\pi} = (\overline{d}_1, \ldots, \overline{d}_n) \), which we will also assume to be nonincreasing. If \( d_n < n - 1 \), then
\[
\overline{d}_{k-n+1} = \overline{d}_{n+t-3} \geq n + t - 3 \geq 2t - 3.
\]
As \( t \geq 3 \), this gives that \( \overline{d}_1, \ldots, \overline{d}_t \geq 2t - 3 > t - 1 \), so \( \overline{\pi} \) is potentially \( K_t \)-graphic by part (b) of Theorem 3.2. In a similar manner, if \( d_t < t - 1 \) then \( \pi \) is potentially \( K_n \)-graphic since \( d_n \geq d_{2n-3} \geq 2n - 3 \). Consequently, we may assume that \( d_n \geq n - 1 \) and \( \overline{d}_t \geq t - 1 \). Additionally, if \( t = 3 \), then \( k \geq 7 \) so \( \overline{\pi} \) is potentially \( K_3 \)-graphic by Theorem 3.1. Hence we may assume that \( t \geq 4 \).

Now, if \( d_{2n} \geq n - 2 \), then part (a) of Theorem 3.2 implies that \( \pi \) is potentially \( K_n \)-graphic. Hence, we may assume that \( d_{2n} \leq n - 3 \), so that \( \overline{d}_{k-2n+1} = \overline{d}_{t-3} \geq (k - 1) - (n-3) \geq 2t - 2 \). As we have that \( \overline{d}_{t-1} \geq t - 1 \), we may apply part (b) of Theorem 3.2 unless \( \overline{d}_{t-2} < t \) or, more specifically since \( \overline{d}_t \geq t - 1 \), if \( \overline{d}_{t-2} = t - 1 \). In this case, however, then \( \overline{d}_{t-(t-2)+1} = d_{2n-1} \geq 2n - 4 \), so we can apply part (b) of Theorem 3.2 to \( \pi \) provided \( d_1 \geq 2n - 3 \). We complete the proof by observing that if this were not the case, we would have \( d_1 = 2n - 4 \) and thus \( \overline{d}_k = k - 1 - (2n - 4) = t - 1 \).

As \( t \geq 4 \), \( k \geq 2t \), \( \overline{\pi} \) is potentially \( K_t \)-graphic by part (a) of Theorem 3.2. \( \square \)

4. \( r_{\text{pot}}(C_n, K_t) \)

Exact values of \( r(C_n, K_t) \) are known for all \( t \leq 7 \), and the conjecture that \( r(C_n, K_t) = (n - 1)(t - 1) + 1 \) for \( n \geq t \geq 3 \) has been outstanding since 1976 [4, 5]; a recent proof for \( n \geq 4m + 2, m \geq 3 \) is given in [13]. In contrast, in this section, we show \( r_{\text{pot}}(C_n, K_t) \) is linear in \( n \) and \( t \) for all \( n \geq 3, t \geq 2 \).

**Theorem 4.1.** For \( t \geq 2 \) and \( n \geq 3 \) with \( t \leq \left\lceil \frac{2n}{3} \right\rceil \), \( r_{\text{pot}}(C_n, K_t) = n + t - 2 \)

**Proof.** The fact that \( r_{\text{pot}}(C_n, K_t) \geq n + t - 2 \) follows from consideration of \( \pi(K_{n-1} \cup K_{t-2}) \), which is unigraphic.

To show the reverse inequality, let \( \pi = (d_1, \ldots, d_k) \) be a graphic sequence with \( k = n + t - 2 \). If \( t = 2 \), the result is immediate, so we assume next that \( t = 3 \) and \( n = 5 \) and order \( \overline{\pi} = (\overline{d}_1, \overline{d}_2, \overline{d}_3, \overline{d}_4, \overline{d}_5, \overline{d}_6) \) to be nonincreasing. By Theorem 3.1, if \( \overline{d}_3 \geq 2 \) then \( \overline{\pi} \) is potentially \( K_3 \)-graphic. We leave it to the reader to check that if \( \overline{\pi} \) is a nonincreasing sequence with six terms that does not satisfy this condition, then \( \pi \) must be potentially \( C_5 \)-graphic.

Hence, we may assume that \( n \geq 6, t \geq 3 \) and \( t \leq \left\lceil \frac{2n}{3} \right\rceil \). If no realization of \( \pi \) contains a cycle, then every realization of \( \pi \) is bipartite, and hence has an independent set of size at least \( \left\lceil \frac{n + t - 2}{2} \right\rceil \). Thus, in the complement of \( G \) we have either a clique on at least \( t \) vertices, or \( \frac{n - t + 2}{2} \leq t - 1 \) which implies that \( t \geq n \). Thus, we can assume that some realization of \( \pi \) contains a cycle. Obviously, if \( G \) contains a cycle of length \( n \), there is nothing to prove, so we assume that no cycle in any realization of \( \pi \) has length \( n \). We proceed by considering the following cases.

**Case 1:** Some realization \( G \) of \( \pi \) contains a cycle of length \( n + 1 \).

Let \( C = v_1v_2\cdots v_nv_{n+1}v_1 \) be a cycle of length \( n + 1 \) in \( G \) and note that \( v_i \neq v_{i+2} \) for any index \( i \), as no realization of \( \pi \) contains an \( n \)-cycle. Furthermore, if \( v_iv_{i+3} \in E(G) \) for some index \( i \), then we can exchange the edges \( v_iv_{i+3} \)
and $v_i +1v_i +2$ for the nonedges $v_i +1v_i +3$ and $v_i v_i +2$ to obtain a graph $G'$ in which
$v_i v_i +2v_i +3 \cdots v_n +1v_1 \cdots v_i$ is a cycle of length $n$. Thus, we can assume that for any
$i$, neither $v_i v_i +2$ nor $v_i v_i +3$ is an edge of $G$.

Consider a vertex $x \in V(G) - C$ with $d = d_C(x)$ maximum. If $d = 0$, we
have either $d_G(x) = 0$ for all $x \in V(G) - C$, or we have $x, y \in V(G) - C$ with
$xy \in E(G)$. In the latter case, we can exchange the edges $v_1 v_2, v_2 v_3$ and $xy$ with
the nonedges $v_1 x, v_3 y$ and $v_1 v_3$ to construct a realization of $\pi$ containing the $n$-
cycle $v_1 v_3 v_4 \cdots v_n +1 v_1$. If, instead, $d_G(x) = 0$ for each vertex in $V(G) - C$, then
$V(G) - C$ along with $v_1, v_3$ and $v_5$ form an independent set of size $t$ unless $v_1 v_5$
is in $E(G)$. Then, however, we can exchange the edges $v_1 v_5, v_2 v_3$ and $v_3 v_4$ for the
nonedges $v_1 v_3, v_3 v_5$ and $v_2 v_4$ to construct a realization of $\pi$ containing the $n$-
cycle $v_1 v_2 v_3 v_5 \cdots v_n +1$.

Thus, we can assume that there is some vertex $x \in V(G) - C$ such that $v_i \in N(x)$
for some index $i$. If $xv_i +1 \in E(G)$, then $G$ contains a cycle of length $n$, so there
exists an index $j$ such that $v_j \in N(x)$ and $v_{j +1} \notin N(x)$. Now, replacing the edges
$xv_j$ and $v_{j +1}v_{j +2}$ with the nonedges $xv_{j +1}$ and $v_{j +1}v_{j +2}$ gives a realization $G'$ of $\pi$
in which $v_j v_{j +2} v_{j +3} \cdots v_n +1 v_1 v_2 \cdots v_j$ is a cycle of length $n$.

**Case 2:** Some realization $G$ of $\pi$ contains a cycle of length $n + 2$.

Let $C = v_1 v_2 \cdots v_n v_{n +2} v_1$ be a cycle of length $n + 2$ in $G$, and note that Case 1
and the assumption that no realization of $G$ contains an $n$-cycle, neither $v_i v_{i +2}$ nor
$v_i v_{i +3}$ are edges of $G$ for any index $i$. If $C$ is not an induced cycle, choose a chord
$v_i v_j \in E(G)$ so that $dist_C(v_j, v_i) > 3$ is minimum. Next, we note that by replacing
the edges $v_i v_{i +1}, v_i v_{i +1}$ and $v_{i +2} v_{i +3}$ with the nonedges $v_i v_{i +2}, v_i v_{i +3}$ and $v_{i +1} v_{j +1}$ we
obtain a realization of $\pi$ in which $v_i v_{i +3} v_{i +4} \cdots v_n +2 v_1 v_2 \cdots v_i$ is a cycle of length
$n$.

Therefore, we can assume that $C$ is an induced $C_{n +2}$ in $G$. Suppose then that
there is a vertex $x$ in $V(G) - C$ such that $xv_i$ is an edge of $G$ for some $i$. Then
either $v_1 v_2 \cdots v_i x v_{i +3} v_{i +4} \cdots v_n +2 v_1$ is a cycle of length $n + 1$, or we can choose an
index $j$ such that $xv_j \in E(G)$ but $xv_{j +1} \notin E(G)$. Then, by exchanging the edges
$v_j x$ and $v_{j +1} v_{j +2}$ with the nonedges $v_j v_{j +2}$ and $xv_{j +1}$ we obtain a realization of
$\pi$ which contains a cycle of length $n + 1$. Furthermore, if there exists any edge
$xy$ with $x, y \notin \{v_1, \ldots, v_n\}$, then replacing the edges $v_1 v_2, v_2 v_3$ and $xy$ with
the edges $v_1 v_3, v_3 x$ and $v_2 y$ yields a realization of $\pi$ where $v_1 v_3 v_4 \cdots v_{n +2} v_1$ is a cycle
of length $n + 1$. Thus $G$ is isomorphic to $C_{n +2} \cup K_{t - 4}$, and as $n + 2 \geq 8$, $\alpha(G) \geq t$.

**Case 3:** Some realization $G$ of $\pi$ contains a cycle of length $m > n + 2$.

Choose a realization $G$ containing a cycle $v_1 v_2 \cdots v_m v_1$ with $m > n + 2$ maximum.
If $v_n \sim v_1$, then the vertices $v_1, v_2, \ldots, v_n$ obviously give the desired subgraph
isomorphic to $C_n$. Similarly, since we can assume $G$ contains no cycle of length $n + 2,
\ v_{n +1} \not\sim v_m$, as this edge would complete a cycle of length $n + 2$. Thus, replacing
the edges $v_m v_1$ and $v_n v_{n +1}$ with the non-edges $v_n v_1$ and $v_{n +1}v_m$ gives a graph $G'$
with degree sequence $\pi$ which contains $C_n$.

**Case 4:** Every realization of $\pi$ has circumference at most $n - 1$. 

As above, let $G$ be a realization of $\pi$ containing a longest cycle $C = v_1v_2 \ldots v_m$ with $m \leq n - 1$ and suppose that $G$ has the maximum circumference amongst all realizations of $\pi$. Let $H$ be the subgraph of $G$ induced by $V(G) - V(C)$.

Claim 1. $H$ is acyclic.

Assume otherwise, and let $x_1x_2 \ldots x_px_1$ be a cycle in $H$. Then, if $v_ix_j \in E(G)$, it follows that neither $v_{i+1}x_j \in E(G)$ nor $v_{i+2}x_{j+1} \in E(G)$. Otherwise, $v_1v_2 \ldots v_ix_jv_{i+1}v_{i+2} \ldots v_mv_1$ or $v_1v_2 \ldots v_ix_{j+1}v_{i+2} \ldots v_mv_1$ is a cycle of length greater than $m$ in $G$, a contradiction. Thus we conclude, without loss of generality, that $v_1 \neq x_2$ and $v_2 \neq x_1$, and by replacing the edges $v_1v_2$ and $x_1x_2$ with $v_1x_2$ and $v_2x_1$, we obtain a realization of $\pi$ with a cycle $v_1x_2x_3 \ldots x_{p-2}x_{p}v_2v_3 \ldots v_mv_1$ of length $m + p$, a contradiction which establishes that $H$ is acyclic.

Claim 2. Either $\Delta(H) \leq 1$ or the only non-trivial component of $H$ is a star.

Assume first that there exist vertices $x$ and $y$ in $H$ such that $d_H(x) \geq 2$ and $d_H(y) \geq 2$. Clearly, we can choose $x$ and $y$ so that $xy \in E(G)$. Since $H$ is acyclic, then for any vertices $x' \in N_H(x) - \{y\}$ and $y' \in N_H(y) - \{x\}$ we have $x' \neq y'$ and $x' \neq x$. As the circumference of $G$ is $m$, there exists an index $i$ such that $v_ix \notin E(G)$ and $v_{i+1}y \notin E(G)$. By replacing $xx'$, $yy'$ and $v_iv_{i+1}$ with $v_ix, v_{i+1}y$ and $x'y'$, we obtain a graph $G'$ realizing $\pi$ containing the $(m + 2)$-cycle $v_1xx'yv_{i+1} \ldots v_mv_1$, a contradiction. Hence $H$ has at most one vertex of degree at least two.

Assume then that $\Delta(H) > 1$, and that $H$ has at least two non-trivial components. Choose $x \in V(H)$ with $d_H(x) > 1$, and choose vertices $x'$ and $x''$ in $N(x)$, and edge $yy'$ in a component of $H$ different from the component containing $x$. We may assume, without loss of generality, that $v_1 \neq x$. If, in addition, $v_2 \neq x$ then $v_1v_2xx''xv_1$ alternates between edges and non-edges in $G$ and can be used to obtain a realization of $\pi$ containing a cycle of length $m + 1 > m$, a contradiction. A similar argument holds if $v_m \neq x$ so both $v_2x$ and $v_mx$ are edges in $G$. Now, if $v_2y \notin E(G)$, then replacing edges $xx'$, $yy'$ and $v_1v_2$ with $v_1x, v_2y$ and $y'x'$ yields a realization of $\pi$ in which $v_1v_2 \cdots v_mv_1$ is a cycle of length $m + 1$, which contradicts the maximality of $m$. We can therefore assume $v_2y \in E(G)$, and by an identical argument, that $v_my' \in E(G)$. But then $v_2 \cdots v_ny'yv_2$ is a cycle of length $m + 1$ in $G$. This last contradiction establishes the claim.

Claim 3. If the only non-trivial component of $H$ is a star, then $\pi \to (C_n, K_t)$.

Let $x$ be the unique vertex in $V(H)$ with $d_H(x) > 1$. Clearly, by the maximality of $m$, $v_ix \in E(G)$ implies that $v_{i+1} \neq x$ for every $i$, and the argument in Claim 2 establishes that we could construct a larger cycle if $x$ has two consecutive non-neighbors on $C$. We conclude that $m$ is even and, without loss of generality, $x \sim v_1$ if and only if $i$ is even. Next, observe that if we can choose an odd index $i$ such that $v_i \sim y$ for some $y \in V(H) - x$, then either $v_1v_2 \cdots v_iyyv_{i+1}v_{i+2} \cdots v_mv_1$ is a cycle of length $m + 1$ in $G$ or, $xy \notin E(G)$. In the latter case, we can exchange the edges $v_2y$ and $xx'$ with the nonedges $v_ix$ and $x'y$ for any $x' \in N_H(x)$ to obtain a realization of $\pi$ in which $v_1v_2 \cdots v_ixv_{i+1}v_{i+2} \cdots v_mv_1$ is a cycle of length $m + 1$. Thus, we conclude that no vertex in $V(H)$ is adjacent to $v_1$ or $v_3$. If $v_1 \sim v_3$, we have the cycle $v_1v_3v_2v_4v_5 \cdots v_mv_1$ of length $m + 1$ in $G$, and if $v_1 \neq v_3$, then $V(H) \cup \{v_1, v_3\} - \{x\}$ is a set of at least $(k - m) + 2 - 1 = (n - m) + t - 1 \geq t$ independent points in $G$, completing the proof of the claim.
As a result of Claims 1 - 3, we may assume that $\Delta(H) \leq 1$. First, consider the case that $\Delta(H) = 0$. Since $|V(H)| = k - m \geq t - 1$, we can assume equality holds, and conclude that $m = n - 1$. Now, either every vertex $v_i$ is incident with at least one vertex of $H$, or we can choose a vertex $v_j$ such that $V(H) \cup \{v_j\}$ is an independent set of size at least $t$. If every $v_i$ has at least one neighbor in $V(H)$, then since $n - 1 > t - 1$, we conclude that some vertex $x \in V(H)$ is adjacent to vertices $v_i$ and $v_j$ with $i < j$, and by the maximality of $m$, $i < j - 1$. We choose $x$ and $v_i$ $v_j$ so that $j - i > 1$ is as small as possible. We now choose $y \in V(H) \cap N(v_{i+1})$, and observe that $y \not\in x$. The minimality of $j - i$ ensures that $v_j \not\sim y$ and $v_{i+1} \not\sim x$. Thus, the graph obtained by replacing edges $v_{i+1}y$ and $v_jx$ with $v_{i+1}x$ and $v_jy$ is a realization of $\pi$ which contains a cycle $v_1v_2\cdots v_tv_{i+1}v_i v_1$ of length $m + 1 = n$.

Therefore, it remains to examine the possibility that $\Delta(H) = 1$. First, we assume that $H$ contains exactly two (necessarily adjacent) vertices of degree one, say $x$ and $y$. We also note that if $v_i x \in E(G)$ for any index $i$, then the maximality of $m$ guarantees that $v_{i+1} \not\sim x$ and $v_{i+2} \not\sim y$. In this case, by exchanging the edges $v_{i+1}v_{i+2}$ and $xy$ with the nonedges $v_{i+1}x$ and $v_{i+2}y$ we obtain a realization of $\pi$ with an independent set of size $|V(H)|$. Since $H$ has exactly one edge and order $k - m$, we conclude that either $H$ has $t$ independent points, or $m$ is one of $n - 1$ or $n - 2$. If $m = n - 2$, then $|V(H)| = t$ and we can use an argument identical to the case where $\Delta(H) = 0$ to show that there is some vertex on $C$ that has no neighbor in $H$, implying the existence of an appropriate independent set. Alternatively, if $m = n - 1$, then $|V(H)| = t - 1$ and we can conclude again that there is some vertex $v_j$ adjacent to no vertex of $H$. Since $m = n - 1 \geq 5$, we can then choose an index $i \neq j, j - 1$ so that replacing the edges $v_iv_{i+1}$ and $xy$ with $v_i x$ and $v_{i+1}y$ gives realization of $\pi$ with $t$ independent points (the vertices in $V(H)$ together with the vertex $v_j$).

Next, assume that $H$ contains $2\ell \geq 4$ vertices of degree one and $p$ isolated vertices and choose $xx', yy' \in E(H)$. If $v_i x \in E(G)$ but $v_i \not\sim x'$, then we can produce a realization of $\pi$ in which $\Delta(H) = 2$ by replacing edges $v_ix, xx'$ and $yy'$ with the nonedges $xy, xy'$ and $v_ix'$. The previous claims then imply $\pi \rightarrow (C_n, K_1)$. Furthermore, we note that by replacing the edges $xx'$ and $yy'$ with the edges $xy$ and $x'y'$ in $H$, we obtain another realization of $\pi$, and as a result the preceding argument immediately implies that $v_i H \in E(G)$. Thus, each vertex $v_i$ adjacent to any vertex $x$ with $d_H(x) = 1$ is adjacent to every vertex of $H$ with degree one. Let $S = \{v_i \mid v_i \sim x$ for some $x$ with $d_H(x) = 1\}$, and let $s = |S|$.

First, suppose that $S = \emptyset$ and that $x_i y_i, 1 \leq i \leq \ell$ are the (disjoint) edges in $H$. If $\ell \geq m$, then we can exchange $x_i y_i$ and the edge $v_i v_{i+1}$ for the nonedges $xx_i$ and $yy_i$. This creates a realization of $\pi$ in which the vertices of $H$ contain an independent set of size $p + \ell + m$. If $p + \ell + m \geq t$ we are done, so suppose otherwise. Then $k - (p + \ell + m) = \ell \geq n - 1$, which implies that $k \geq 2n - 2$, a contradiction. Hence we may assume that $\ell < m$, which again allows us to exchange edges in $H$ and edges on $C$ to create a realization of $\pi$ in which $H$ contains an independent set of size $p + 2\ell$. As $k = n + t - 2$ and $m \leq n - 1$, this implies that, in fact, $p + 2\ell = t - 1$ and $m = n - 1$. Now, as above, if every vertex $v_i$ has a neighbor in $H$, we can construct a realization of $G$ containing a longer cycle, so there is some vertex $v_j$ with no neighbor in $H$. As $t \leq \left\lceil \frac{2n}{3} \right\rceil$ and $\ell \leq \frac{t - 1}{2}$, we conclude that $\ell \leq n - 3 = m - 2$, so we can use edges in $E(C)$ that are not incident to $v_j$ to
contains a cycle of length \( n \) and create a realization of \( \pi \) in which \( V(H) \cup \{v_j\} \) forms an independent set of size \( t \), as desired.

Thus, we now assume that \( S \neq \emptyset \). Observe that for every \( v_i \in S \), the maximality of \( m \) implies that \( v_{i+2} \) and \( v_{i-2} \) are not adjacent to any vertex of \( V(H) \) with degree one. Additionally, the vertices \( v_{i+1} \) and \( v_{i-1} \) are not adjacent to any vertex \( x \in V(H) \), as we would either have a cycle of length \( m + 1 \) if \( d_H(x) = 1 \), or, if \( d_H(x) = 0 \), we can choose \( yy' \in E(H) \) and replace \( v_{i+1}x \) with \( v_{i+1}y \) and \( v_{i}x \) to obtain a realization in which \( v_1 \cdots v_jyv_{j+1} \cdots v_nv_1 \) is a cycle of length \( m + 1 \) for \( j = i, i - 1 \).

Furthermore, we note that each edge \( v_iv_j \) of the graph induced by \( C - S \), and any edge \( xx' \in E(H) \) can be replaced with the edges \( v_ix \) and \( v_ix' \) to obtain a realization of \( \pi \) with fewer edges in the graph induced by \( V(H) \). Since each edge of the cycle not incident with a vertex of \( S \) in \( C - S \) and \( dist_C(v_i, v_j) \geq 3 \) for any distinct \( v_i \) and \( v_j \) in \( S \), there are at least \( m - 2s \) edges in this graph. If \( m - 2s > t \), then we can eliminate every edge of \( H \) without using an edge incident with \( v_{i+1} \) for some \( v_i \in S \). This gives a realization of \( \pi \) in which \( V(H) \cup \{v_{i+1}\} \) is a collection of at least \( t \) independent vertices. Thus, we can assume that \( m - 2s \leq t \), and by using each of these \( m - 2s \) edges of the cycle to eliminate an edge of \( H \), we conclude that there is a realization of \( \pi \) with an independent set of size \( \ell + p \) with \( m - 2s \). So either \( \pi \to (C_n, K_t) \) or

\[
\ell + p + m - 2s \leq t - 1. \tag{1}
\]

Now, let \( S^+ = \{v_{i+1} \mid v_i \in S\} \), and suppose first that there is some \( v \in S^+ \) and \( z \in V(H) \) such that \( v \sim z \) and \( d_H(z) = 0 \). Then, for any edge \( xy \) in \( H \) we could exchange the edges \( vz \) and \( xy \) for the nonedges to obtain a realization of \( \pi \). This results in \( x \) having consecutive neighbors on \( C \), a contradiction to the maximality of \( m \). We also claim that \( S^+ \) forms an independent set in \( G \). If not, then there are adjacent vertices \( v_{i+1} \) and \( v_{j+1} \) in \( S^+ \), which implies that for any edge \( xy \) in \( H \), \( v_ixyv_jv_{j+1} \cdots v_{i+1}v_{j+1}v_{j+2} \cdots v_i \) is an \((m + 2)\)-cycle in \( G \), a contradiction. Thus, \( S^+ \) along with the \( p \) isolates in \( H \) and one vertex from each edge in \( H \) comprise an independent set in \( G \). Thus, either \( \pi \to (C_n, K_t) \) or

\[
\ell + p + s \leq t - 1. \tag{2}
\]

Summing (1) and (2) gives \( 2 \ell + 2p + m - s \leq 2t - 2 \). Now, since \( 2\ell + p = |V(H)| = n + t - 2 - m \), we have \((n - s) + p \leq t \). Note that \( s \leq \frac{4n}{3} \leq \frac{4n - 4}{3} \), and hence \( \frac{2n + 1}{3} + p \leq t \), contradicting \( t \leq \left\lceil \frac{2n}{3} \right\rceil \) and completing the proof. \( \square \)

**Theorem 4.2.** For \( t \geq 3 \) and \( n \geq 4 \) with \( t > \left\lceil \frac{4n}{3} \right\rceil \),

\[
r_{pot}(C_n, K_t) = 2t - 2 + \left\lceil \frac{n}{3} \right\rceil.
\]

**Proof.** The lower bound stems from Theorem 2.2 and/or consideration of \( \pi(K_{\left\lceil \frac{4n}{3} \right\rceil - 1}) \lor (t - 1)K_2 \), which is unigraphic. To show the reverse inequality, let \( \pi = (d_1, \ldots, d_k) \) be graphic with \( k = 2t - 2 + \left\lceil \frac{4n}{3} \right\rceil \). If no realization of \( \pi \) contains a cycle, then every realization of \( \pi \) is bipartite, and hence has an independent set of size at least \( t \). Thus, we can assume that some realization of \( \pi \) contains a cycle. Obviously, if \( G \) contains a cycle of length \( n \), there is nothing to prove, so we assume that no cycle
in any realization of $\pi$ has length $n$. Due to the similarities between this proof and that of Theorem 4.1, we only sketch the following cases, which suffice to complete the proof.

**Case 1:** Some realization $G$ of $\pi$ contains a cycle of length $n + 1$.

Let $C = v_1v_2 \cdots v_nv_{n+1}v_1$ be a cycle of length $n + 1$ in $G$. Just as in our proof of Theorem 4.1, this requires that $d(x) = 0$ for every $x \in V(G) - V(C)$. Then either $v_1 \sim v_3$ and $v_1v_3v_4 \cdots v_{n+1}v_1$ is a cycle of length $n$, or $V(G) - \{v_2, v_4, \ldots, v_{n+1}\}$ is a set of $k - (n - 1) = 2t - 1 - \left\lfloor \frac{2m}{3} \right\rfloor \geq t$ independent vertices.

**Case 2:** Some realization $G$ of $\pi$ contains a cycle $v_1v_2 \cdots v_{n+2}v_1$.

Again, the arguments in Case 2 of Theorem 4.1 imply that $G = C_{n+2} \cup \overline{K_{k-n-2}}$ and since $n \geq 4$, this graph has an independent set of size at least $k - n + 1 \geq t$.

**Case 3:** Some realization of $\pi$ has circumference $m > n + 2$.

The proof is identical to the proof of Case 3 of Theorem 4.1.

**Case 4:** Every realization of $\pi$ has circumference $m \leq n - 1$.

Let $G$ be a realization of $\pi$ containing a longest cycle $C = v_1v_2 \cdots v_m$ with $m \leq n - 1$ and suppose that $G$ has the maximum circumference amongst all realizations of $\pi$. Let $H$ be the subgraph of $G$ induced by $V(G) - V(C)$ and note that $H$ has order at least $k - (n - 1) \geq t$ so that $H$ contains at least one edge. Following the argumentation in the proof of Theorem 4.1 we conclude that $H = \ell K_2 \cup pK_1$ for some integers $p$ and $t$. Proceeding in the same manner, we define the set $S = \{v_i \mid v_i x \in E(G) \text{ for some vertex } x \text{ with } \deg_H(x) = 1\}$, and note that $v_i \in S$ implies that none of $v_{i+1}, v_{i+2} \in S$. Thus, the graph $V(C) - S$ has at least $m - 2|S|$ edges and each edge of this graph can be paired with an edge of $H$. Replacing these edges with two vertex disjoint non-edges in the induced $2K_2$ gives a realization of $\pi$ with one fewer edge in the graph induced by $V(H)$. Thus, we have some realization of $\pi$ with an independent set of size at least $\ell + p + m - 2|S|$, so either $\pi \to (C_n, K_t)$ or $\ell + p + m - 2s \leq t - 1$. Since $|V(H)| = 2\ell + p = 2t - 2 + \left\lfloor \frac{2m}{3} \right\rfloor - m$, we must have

$$2(\ell + p + m - 2|S|) \leq 2t - 2$$

Thus, there is no realization of $\pi$ with an independent set of size at least $\ell + p + m - 2|S|$, so either $\pi \to (C_n, K_t)$ or $\ell + p + m - 2s \leq t - 1$.

An easy case analysis then shows that the left hand side is strictly positive except when $p = 0$, $m = n - 1$, $|S| = \left\lfloor \frac{2m}{3} \right\rfloor$ and $n \equiv 1 \pmod{3}$. However, this in turn requires $k = 2t - 2 + \frac{n+2}{3}$, and $|V(H)| = 2\ell = 2t - 2\frac{n+1}{3}$. As $n \equiv 1 \pmod{3}$, $2n+1$ is odd, a contradiction which completes the proof. \hfill \Box

The techniques developed in the proofs of Theorems 4.1 and 4.2 also allow us to determine $r_{pot}(P_n, K_t)$. In fact, $r_{pot}(P_n, K_t)$ and $r_{pot}(C_n, K_t)$ differ by at most one. Note, however, that $r(P_n, K_t) = (n - 1)(t - 1) + 1$ [14], which is the conjectured value for $r(C_n, K_t)$ for all $n \geq t \geq 3$.

**Theorem 4.3.** For $n \geq 6$ and $t \geq 3$,
\[ r_{pot}(P_n, K_t) = \begin{cases} 
  n + t - 2 & t \leq \left\lfloor \frac{2n}{3} \right\rfloor 
  2t - 2 + \left\lfloor \frac{4n}{3} \right\rfloor & t > \left\lfloor \frac{2n}{3} \right\rfloor. 
\end{cases} \]

**Proof.** When \( t \leq \left\lfloor \frac{2n}{3} \right\rfloor \), we note that \( r_{pot}(P_n, K_t) \leq r_{pot}(C_n, K_t) = n + t - 2 \), and equality is established via Corollary 2.3 and/or by observing that \( K_{n-1} \cup \overline{K_{t-2}} \) contains neither \( P_n \) nor \( K_t \).

Therefore, we may assume \( t > \left\lfloor \frac{2n}{3} \right\rfloor \). Theorem 2.2 establishes that \( r_{pot}(P_n, K_t) \geq 2t - 2 + \left\lfloor \frac{4n}{3} \right\rfloor \) and if \( n \equiv 0 \mod 3 \), we observe that \( r_{pot}(P_n, K_t) \leq r_{pot}(C_n, K_t) = 2t - 2 + \frac{4}{3} \), establishing the result.

Finally, for \( n \not\equiv 0 \mod 3 \), we note that by Theorem 4.1, every degree sequence \( \pi \) with \( k \geq 2t - 2 + \left\lfloor \frac{4n}{3} \right\rfloor \) terms has a realization \( G \) which contains either a subgraph isomorphic to \( C_{n-2} \) or an independent set of cardinality \( t \). If the latter, \( \pi \to (P_n, K_t) \), and if the former, we note that there are at least \( 2t - 2 + \left\lfloor \frac{4n}{3} \right\rfloor - (n - 2) = 2t - \left\lfloor \frac{2n}{3} \right\rfloor \geq t \) vertices in \( V(G) - V(C_n-2) \). Since we are assuming \( \alpha(G) < t \), there is at least one edge \( xy \in E(G) \) with \( x, y \notin V(C_n-2) \). Now, without loss of generality, either \( xy \in E(G) \) and \( G \) contains a subgraph isomorphic to \( P_n \), or we can select any edge \( v_i v_{i+1} \in E(C_{n-2}) \) and exchange the edges \( v_i v_{i+1} \) and \( xy \) with the non-edges \( v_i x \) and \( v_{i+1} y \) to obtain a realization of \( \pi \) which contains a path of order \( n \). \( \square \)

5. Conclusion

Our results in this paper deal only with a “2-color” Ramsey-type parameter for degree sequences; we are also able to define a multicolored version of \( r_{pot} \). Let \( \pi_1, \ldots, \pi_k \) be graphic sequences, with \( \pi_i = (d_1^{(i)}, \ldots, d_n^{(i)}) \) for all \( i \) (they need not be monotone). Then, as defined in [2], \( \pi_1, \ldots, \pi_k \) pack if there exist edge-disjoint graphs \( G_1, \ldots, G_k \), all with vertex set \( \{v_1, \ldots, v_n\} \), such that

\[ d_{G_i}(v_j) = d_j^{(i)} \]

for each \( i \), and

\[ d_{G_1 \cup \cdots \cup G_k}(v_j) = \sum_{i=1}^n d_j^{(i)}. \]

In traditional graph packing, the vertex sets of the graphs in question may be permuted prior to their embedding into \( K_n \). Note that when packing degree sequences, the ordering of the terms in \( \pi_1, \ldots, \pi_k \) remains fixed.

With this definition in mind, we define \( r_{pot}(G_1, \ldots, G_k) \) to be the smallest integer \( n \) such that for any collection of \( n \)-term graphic sequences \( \pi_1, \ldots, \pi_k \) that sum, termwise, to \( n - 1 \) and pack, there exist edge disjoint graphs \( F_1, \ldots, F_k \) all with vertex set \( \{v_1, \ldots, v_n\} \), such that \( d_{F_i}(v_j) = d_j^{(i)} \) for all \( i, j \) and also that \( F_i \) contains \( G_i \) as a subgraph for some \( i \). As is the case with the two-color version, the multicolor potential-Ramsey number is bounded from above by the classical Ramsey number, but the added challenges inherent in degree sequence packing make the determination of \( r_{pot}(G_1, \ldots, G_k) \) for \( k \geq 3 \) both interesting and significantly more difficult than the \( k = 2 \) case.
REFERENCES

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