Math 450 Combinatorics Handout

The Twelvefold Way

We wish to count the number of functions $f : N \to K$, where $|N| = n$ and $|K| = k$, with the additional restrictions that f might be injective or surjective, and that the elements of N and K may be distinguishable or indistinguishable.

A way of interpreting such functions is of putting n balls into k boxes; the function f says that ball i goes into box $f(i)$. The balls and boxes may or may not be labeled. For an injective function (*i.e.*, one-to-one), no two balls are put into the same box. For a surjective function (*i.e.*, onto), no box is empty. A bijection is a function that is both injective and surjective.

"balls"	"boxes"	conditions on f			
N	K	none	injective	surjective	bijective
dist.	dist.	k^n	$P(k,n)$	$k!S(n,k)$	$\begin{cases} k!, & \text{if } n = k \\ 0, & \text{if } n \neq k. \end{cases}$
indist.	dist.	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{n-k}$	$\begin{cases} 1, & \text{if } n = k \\ 0, & \text{if } n \neq k. \end{cases}$
dist.	indist.	$\sum_{t=0}^{k} S(n,t)$	$\begin{cases} 1, & \text{if } n \leq k \\ 0, & \text{if } n > k. \end{cases}$	$S(n,k)$	$\begin{cases} 1, & \text{if } n = k \\ 0, & \text{if } n \neq k. \end{cases}$
indist.	indist.	$\sum_{t=1}^{k} p(n,t)$	$\begin{cases} 1, & \text{if } n \leq k \\ 0, & \text{if } n > k. \end{cases}$	$p(n,k)$	$\begin{cases} 1, & \text{if } n = k \\ 0, & \text{if } n \neq k. \end{cases}$

$S(n,k)$ is the Stirling number of the second kind, defined by:

$$\begin{aligned} S(0,0) &= 1, \\ S(n,0) &= 0, \text{ for } n \geq 1, \\ S(n,k) &= 0, \text{ if } n < k, \\ S(n,k) &= kS(n-1,k) + S(n-1,k-1), \text{ for } n \geq k \geq 1. \end{aligned}$$

The Bell number B_n is the number of ways of putting n balls into nonempty boxes. However, the *number* of boxes is not specified. Thus, $B_n = \sum_{t=1}^{n} S(n,t)$. Alternatively, B_n is the number of partitions of $[n]$ into nonempty parts.

$p(n,k)$ is the number of partitions of the integer n into k (nonempty) parts. $\sum_{t=1}^{k} p(n,t)$ is the number of partitions of n into at most k nonempty parts (or, alternatively, partitions of n into exactly k parts, some of which may be empty). $p_n = \sum_{k=1}^{n} p(n,k)$ is the number of partitions of n into (any number of) nonempty parts.