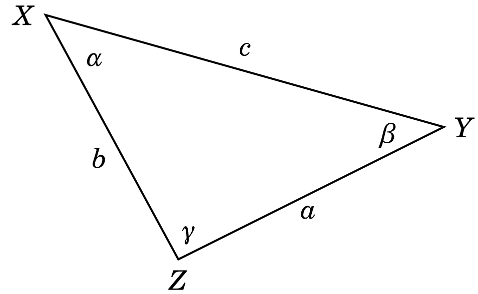
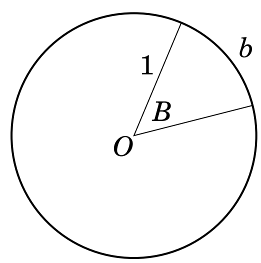
Spherical Law of Cosines

We will develop a formula simlar to the Euclidean Law of Cosines. Let XYZ be a triangle, with angles α, β, γ and opposite side lengths a, b, c as shown in the figure.

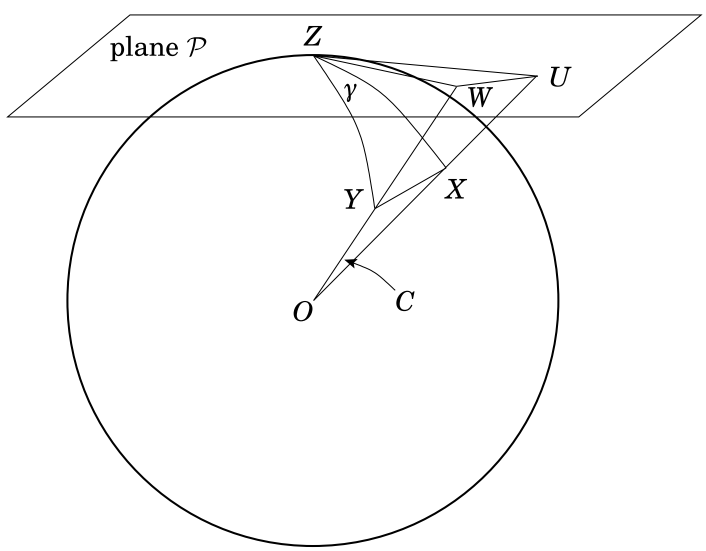


Theorem (Law of Cosines for the Euclidean plane)**.** $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

Now assume that our triangle XYZ is on a unit sphere, with angles α, β, γ and side lengths a, b, c. Each side is a portion of a great circle, and (since the sphere has unit radius), the side length has the same measure as the angle (in radians) at the center O of the sphere. Let A, B, C be the central angles of the sides a, b, c.



We assume that the point Z is at the north pole of our sphere. Consider the plane $\mathcal{P}$ tangent to the sphere at point Z. Extend a ray from the center O of the sphere through point X to the plane $\mathcal{P}$; call U the point of intersection between the ray and the plane. Similarly, extend a ray from O through Y to $\mathcal{P}$; call W the point of intersection.



Next calculate the length of the segment UW in the plane $\mathcal{P}$ using the Euclidean Law of Cosines applied to the triangle UZW. To do this, you need the length of the segments ZU and ZW. Note that the angle of UZW is γ.

Finally, apply the Euclidean Law of Cosines to the triangle OWU. Note that the angle of UOW is C, which is our goal. You will need the length of the segments OU and OW. Simplify the expression (the Pythagorean identity $(\tan\theta)^2 + 1 = (\sec\theta)^2$ may be helpful).

Theorem (Spherical Law of Cosines)**.** $\cos C = (\cos A)(\cos B) + (\sin A)(\sin B)\cos\gamma$.

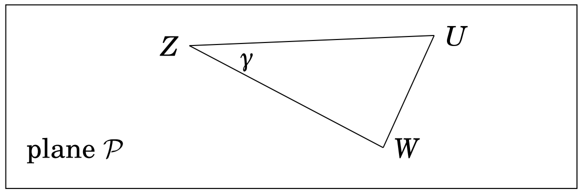
Application. Find the distance

1. from Seattle (48°N, 2°E) to Paris (48°N, 122°W). What is the distance if traveling due east?
2. from Lincoln, NE (40°N, 96°W) to Sidney, Australia (34°S, 151°E).

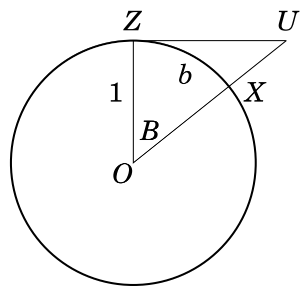
Question 3. What is the "Pythagorean theorem" for right triangles on the sphere? Does it work for triangles with more than one right angle?

Question 4. We know that Side-Angle-Side is a congruence relation on the sphere. Suppose that we know that (on a unit sphere) a triangle has a side of length 1, a side of length 1.5, and an included angle of 60°. What is the length of the third side and the measure of the other two angles?

We next calculate the length of the segment UW in the plane $\mathcal{P}$ using the Euclidean Law of Cosines applied to the triangle UZW.



First, we need the length of the segments ZU and ZW. The length of ZU can be calculated by looking at the cross section of the sphere containing O, Z, and U.



Hence, the length of ZU is $\tan B$ and similarly, the length of ZW is $\tan A$. Thus,

$$\begin{aligned}(\text{length of } WU)^2 &= (\text{length of } ZU)^2 + (\text{length of } ZW)^2 - 2\,(\text{length of } ZU)\,(\text{length of } ZW)\cos\gamma \\ &= (\tan B)^2 + (\tan A)^2 - 2\,(\tan B)\,(\tan A)\cos\gamma.\end{aligned}$$

Finally, we apply the Euclidean Law of Cosines to the triangle OWU. Note that the angle of UOW is C, which is our goal. First note that the length of OU is $\sec B$ from the above figure. Similarly, the length of OW is $\sec A$. Thus,

$$\begin{aligned}(\text{length of } WU)^2 &= (\text{length of } OU)^2 + (\text{length of } OW)^2 - 2\,(\text{length of } OU)\,(\text{length of } OW)\cos C \\ &= (\sec B)^2 + (\sec A)^2 - 2\,(\sec B)\,(\sec A)\cos C.\end{aligned}$$

Equating the two expressions for (length of $WU)^2$, we obtain

$$\begin{aligned}(\tan B)^2 + (\tan A)^2 - 2\,(\tan B)\,(\tan A)\cos\gamma &= (\sec B)^2 + (\sec A)^2 - 2\,(\sec B)\,(\sec A)\cos C \\ &= (\tan B)^2 + (\tan A)^2 + 2 - 2\,(\sec B)\,(\sec A)\cos C,\end{aligned}$$

where we used the Pythagorean identity $(\tan\theta)^2 + 1 = (\sec\theta)^2$ for the last step. Simplifying, we have

$$(\tan B)\,(\tan A)\cos\gamma = (\sec B)\,(\sec A)\cos C - 1,$$

and cross-multiplying by $(\cos A)(\cos B)$, we obtain

$$(\sin A)\,(\sin B)\cos\gamma = \cos C - (\cos A)\,(\cos B).$$

Theorem (Spherical Law of Cosines)**.** $\cos C = (\cos A)\,(\cos B) + (\sin A)\,(\sin B)\cos\gamma.$

Calculation of Distance on the Earth using Latitude and Longitude

Suppose that we wish to calculate the distance between two points on the surface of the Earth. Let the starting point S have latitude ϕ_S and longitude λ_S, and the finish point have latitude ϕ_F and longitude λ_F. We consider the triangle SNF, where N is the North Pole. The angle SNF is $\gamma = \lambda_F - \lambda_S$. The angular length of NS is $A = 90° - \phi_S$, and the angular length of NF is $B = 90° - \phi_F$. Thus, by the Spherical Law of Cosines, the angular distance C from S to F satisfies

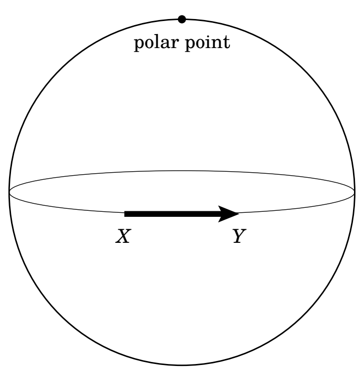
$$\cos C = (\cos A)(\cos B) + (\sin A)(\sin B)\cos\gamma.$$

The actual distance from S to F can be computed using the radius of the Earth, 6371 km.

Example. Seattle is at 47.6°N, 122.3°W, and Paris is at 48.8°N, 2.7°E. We compute $\gamma = 122.3 + 2.7 = 125°$, $A = 90 - 47.6 = 42.4$, and $B = 90 - 48.8 = 41.2°$. Thus, $\cos C = .3009$, $C = 1.2652$ (in radians), and the distance is $C(6371) = 8061$ km.

Spherical Law of Cosines for Angles

We know that Angle-Angle-Angle is a congruence relation for spherical geometry. We can use the Spherical Law of Cosines for Angles to prove this.
We again assume that our triangle XYZ is on a unit sphere, with angles α, β, γ and side lengths a, b, c, with corresponding central angles A, B, C.
Since each line segment on the sphere a part of a great circle, we define the *polar point* of a line segment to be the center of the great circle it lies on. Since there are two possible choices, we make the choice well defined using the right-hand rule: when moving from X to Y along the line segment, the center should be on the left.



The *dual triangle* of XYZ is the triangle whose vertices are the polar points of the sides of XYZ. Draw several triangles on the Lénárt sphere and their duals.

Proposition. *Given a triangle with side lengths A, B, C (measured as central angles) and opposite angles α, β, γ, the dual triangle formed from the polar points has side lengths $\pi-\alpha, \pi-\beta, \pi-\gamma$ (measured as central angles) and angles $\pi - A, \pi - B, \pi - C$. Moreover, the dual of the dual triangle is the original triangle.*

Applying the Spherical Law of Cosines to the dual triangle, we obtain

Theorem (Spherical Law of Cosines for Angles)**.** $\cos\gamma = (\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\beta)\cos C.$