On the Domination Number
of Products of Graphs: I

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ABSTRACT

In this paper we consider a problem of Cockayne, to
determine a relationship between the domination number of a
graph product versus the product of the domination numbers.
For some very special cases we show, in fact, that

\[ \sigma(G \times H) \succeq \sigma(G) \cdot \sigma(H), \]

where \( \sigma(F) \) is the domination number of the graph \( F \). This paper supports the conjecture that statement (1) is true for all
graphs \( G \) and \( H \).

Introduction. In this paper, we discuss finite undirected simple graphs.
For any undefined terms see [1]. For any graph \( G \), we denote by \( V(G) \) and
\( E(G) \), the vertex and edge set of \( G \) respectively. A subgraph \( H \subseteq G \),
denoted \( H \subseteq G \) is a graph with \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). The
product of two graphs, \( G \times H \), is a graph with \( V(G \times H) = V(G) \times V(H) \) and \((g_1, h_1), (g_2, h_2) \in E(G \times H) \) if and only
if either \( g_1 = g_2 \) and \( h_1 h_2 \in E(H) \) or \( g_1 g_2 \in E(G) \) and \( h_1 = h_2 \). A sub-
set \( S \subseteq V(G) \) is a dominating set of \( G \) if for every \( x \in V(G) - S \), there
is at least one vertex \( y \in S \) such that \( xy \in E(G) \). Finally, the domination number of a graph \( G \), denoted \( \sigma(G) \), is the order of the smallest dom-
inating set.

There has been a good deal of research done on \( \sigma(G) \), see [2] and [3].
It is the purpose of this paper to consider one question given by Cockayne
in [2].

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Conjecture. \( \sigma(G \times H) \geq \sigma(G) \cdot \sigma(H) \).

In this paper we present some bounds for the domination number of a graph. We go on to prove the question for a number of classes of graphs; Paths, Cycles and graphs with dominating numbers half their order. We conclude the paper with the determination of \( \sigma(P_m \times P_n) \) for \( m = 2, 3, \) and \( 4 \) and all \( n \).

Before proceeding we state a few necessary results. We will employ the following notation: \( [x]([x]) \) will denote the smallest (largest) integer greater (less) than or equal to \( x \). The following Theorem was given in [5].

**Theorem A.** If \( C_n \) is the cycle of order \( n \), then

\[
\sigma(C_n) = \left\lceil \frac{n}{3} \right\rceil.
\]

Also, if \( P_n \) is the path of order \( n \), then

\[
\sigma(P_n) = \left\lceil \frac{n}{3} \right\rceil.
\]

For the next result, we define property (*) as the following:

(*) The vertices of a graph \( G \) can be partitioned into two sets, \( V_1 = \{v_1, v_2, \ldots, v_n\} \) and \( V_2 = \{u_1, u_2, \ldots, u_n\} \) with only a matching between \( V_1 \) and \( V_2 \) (these are the only edges between these vertices) and satisfying \( <V_1> = K_n \) and \( <V_2> \) is connected. The following was proved in [4].

**Theorem B.** A connected graph \( G \) of order \( 2n \) has \( \sigma(G) = n \) if, and only if, either \( G = C_4 \) or \( G \) satisfies (*).

**General Bounds.** Throughout the remainder of this paper we will refer to the maximum degree of a graph \( G \) as \( \Delta(G) \). The \( G \) will be dropped \( (\Delta(G) = \Delta) \) when no confusion will result. We begin by proving the following useful lower bound on \( \sigma(G) \).

**Lemma 1.** For every graph \( G \) of order \( n \),

\[
\sigma(G) \geq \frac{n}{\Delta + 1}.
\]

**Proof.** Let \( G \) be a graph of order \( n \) and let \( S \) be a minimum dominating set for \( G \). Then

\[
|S| \cdot \Delta(G) \geq \sum_{v \in V(G) - S} d(v) \geq |V(G) - S| = n - |S|.
\]

Thus with \( \sigma(G) = |S| \), the result follows. \( \Box \)
Corollary 2. If \( G \) is a graph of order \( m \) then
\[
\sigma(C_n \times G) \geq \frac{mn}{\Delta(G)+3}.
\]

\textbf{Proof.} Clearly, \( \Delta(C_n \times G) = \Delta(G) + 2 \). Invoking Lemma 1 gives the result. \( \Box \)

\textbf{Theorem 3.} For graphs \( G \) and \( H \),
\[
\sigma(G \times H) \geq \frac{|V(H)|}{\Delta(H)+1} \cdot \sigma(G).
\]

\textbf{Proof.} Let \( G \) and \( H \) be graphs with order \( m \) and \( n \) respectively. It follows that in \( G \times H \) there are \( n \) canonical disjoint copies of \( G \); label these copies \( G_1, G_2, \ldots, G_n \). Let \( D \) be a minimum dominating set of \( G \times H \) and define \( D_i = G_i \cap D \). Let \( S = \{g_i \in V(G_i) \text{ not dominated by a vertex in } D_i\} \). Clearly, \( |V(G_i) \cap S| \geq \sigma(G) - |D_i| \), for all \( i \), for otherwise there would exist a subset of vertices of \( G \); \( D_i \cup (V(G_i) \cap S) \), of order strictly less than \( \sigma(G) \) which would dominate \( G \). This would be a contradiction. Also, since the vertices in \( S \) must be dominated by vertices in \( D \), it follows that
\[
\Delta(H) \cdot |D| \geq |S| \geq \sum_{i=1}^{n} \sigma(G) - |D_i| + n \cdot \sigma(G) - |D|.
\]

But this implies that
\[
(\Delta(H)+1)|D| \geq n \cdot \sigma(G)
\]

which implies
\[
|D| \geq \frac{n}{\Delta(H)+1} \cdot \sigma(G).
\]

Consequently, \( \sigma(G \times H) \geq \frac{n}{\Delta(H)+1} \cdot \sigma(G) \). \( \Box \)

Note, it also follows for paths.

\textbf{Corollary 4.} \( \sigma(C_{3n} \times G) \geq \sigma(C_{3n}) \cdot \sigma(G) \).

\textbf{Proof.} By Theorem 3 it follows that
\[
\sigma(C_{3n} \times G) \geq \frac{3n}{3} \cdot \sigma(G) = n \cdot \sigma(G).
\]

The result follows since \( \sigma(C_p) = \lfloor \frac{p}{3} \rfloor \), by Theorem A. \( \Box \)
We now are ready to show that Cockayne's conjecture is true for all cycles. Note, it also follows for paths.

**Theorem 5.** If \( m, n \geq 2 \) then \( \sigma(C_m \times C_n) \geq \sigma(C_m) \cdot \sigma(C_n) \).

**Proof.** By Lemma 1, \( \sigma(C_m \times C_n) \geq \frac{mn}{5} \). It is a simple exercise to show that \( \frac{mn}{5} \geq \left\lfloor \frac{m}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor \) for \( m \) and \( n \geq 3 \). Hence \( \sigma(C_m \times C_n) \geq \left\lfloor \frac{m}{3} \right\rfloor \cdot \left\lfloor \frac{n}{3} \right\rfloor = \sigma(C_m) \cdot \sigma(C_n) \). \( \square \)

We conclude this section with another class of graphs for which the conjecture is true.

**Theorem 6.** If \( G \) is a connected graph of order \( 2n \) with \( \sigma(G) = n \) and \( G \neq C_4 \), then for any graph \( H \),

\[ \sigma(G \times H) \geq \sigma(G) \cdot \sigma(H). \]

**Proof.** Let \( G \) and \( H \) be as in the statement of the theorem. By Theorem B, the vertices of \( G \) can be partitioned into two sets \( V_1 = \{v_1, \ldots, v_n\} \) and \( V_2 = \{u_1, \ldots, u_n\} \) such that \( \langle V_1 \rangle = K_n \), \( \langle V_2 \rangle \) is connected, and there is only a matching between \( V_1 \) and \( V_2 \). We assume that the vertices have been labelled so that, for each \( i \), \( v_i u_i \in E(G) \).

Let \( D \) be a minimum dominating set of \( G \times H \). For each \( i \), set \( W_i = \{ x \in V(H) : \langle v_i, x \rangle \in D \text{ or } \langle u_i, x \rangle \in D \} \). We show that \( W_i \) dominates \( H \). Let \( y \in V(H) - W_i \). Since \( u_i \) is the only neighbor of \( v_i \) in \( G \) and since \( \langle v_i, y \rangle \) and \( \langle u_i, y \rangle \) are not in \( D \), there is an \( x \in V(H) \) satisfying \( \langle v_i, x \rangle \in D \) and \( \langle v_i, x \rangle \) is adjacent to \( \langle v_i, y \rangle \) in \( G \times H \). This implies that \( x \in W_i \) and \( x \) is adjacent to \( y \) in \( H \). Thus, \( W_i \) dominates \( H \) and so \( |W_i| \geq \sigma(H) \). Hence

\[ \sigma(G \times H) = |D| \geq \left| \bigcup_{i=1}^{n} W_i \right| \geq n \sigma(H) = \sigma(G) \cdot \sigma(H). \square \]

**Exact Results.** In this section we determine exactly \( \sigma(G \times H) \) for a few special cases.

**Theorem 7.** \( \sigma(P_2 \times P_n) = \left\lfloor \frac{n+1}{2} \right\rfloor. \)
**Proof.** Consider $P_2 \times P_n$ as two canonical copies of $P_n$ with vertices labeled $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with, for each $i, x_i, y_i$ the only edges between the two paths. First, to show $\sigma(P_2 \times P_n) \geq \lceil \frac{n+1}{2} \rceil$, if $n$ is odd, let $D$ consist of those vertices $x_i, y_i$ where $i = 1 \pmod{4}$ and $j = 3 \pmod{4}$. If $n$ is even, let $D$ consist of the same vertices together with $x_n$. It is an easy matter to check that in each case $D$ dominates $P_2 \times P_n$ and that $|D| = \lceil \frac{n+1}{2} \rceil$. Thus $\sigma(P_2 \times P_n) \leq |D| = \lceil \frac{n+1}{2} \rceil$.

To show that $\sigma(P_2 \times P_n) \geq \lceil \frac{n+1}{2} \rceil$ we proceed by induction. Clearly, the theorem is true for $n = 1, 2$ and 3. Hence, let $n \geq 3$ be an integer for which $\sigma(P_2 \times P_k) \geq \lceil \frac{k+1}{2} \rceil$ for all $k, 1 \leq k \leq n$, and thus, by the argument above, equality holds for $k$. Further, suppose that $\sigma(P_2 \times P_{n+1}) < \lceil \frac{n+2}{2} \rceil$. Then $\lceil \frac{n+1}{2} \rceil = \sigma(P_2 \times P_n) \leq \sigma(P_2 \times P_{n+1}) < \lceil \frac{n+2}{2} \rceil$. This implies that $n$ is odd and $\sigma(P_2 \times P_{n+1}) = \lceil \frac{n+1}{2} \rceil$. Let $D$ be a minimum dominating set of $P_2 \times P_{n+1}$.

Suppose first that $x_1 \notin D$. Then by removing $x_1$ and, if necessary, replacing $y_2$ by $y_3$ in $D$, the resulting set $D'$ would dominate the induced subgraph

$$\langle \{x_3, y_3, x_4, y_4, \ldots, x_{n+1}, y_{n+1}\} \rangle,$$

which is $P_2 \times P_{n-1}$. Thus we would have, since $n$ is odd,

$$\lceil \frac{n}{2} \rceil = \sigma(P_2 \times P_{n-1}) \leq |D'| = \lceil \frac{n+1}{2} \rceil.$$

This contradiction implies $x_1 \notin D$. Similarly, $y_1 \notin D$. Since each vertex must be dominated by $D$, it must follow that both $x_2$ and $y_2 \notin D$. We also note that neither $x_3$ nor $y_3$ is in $D$. If, for instance $x_3 \in D$, replacing both $x_2$ and $y_2$ by $y_1$ would give a smaller dominating set for $P_2 \times P_{n+1}$ than the minimum set we chose. Hence the set $D^* = D - \{x_2, y_2\}$ dominates the induced subgraph

$$\langle \{x_4, y_4, x_5, y_5, \ldots, x_{n+1}, y_{n+1}\} \rangle,$$

which is $P_2 \times P_{n-2}$. Thus, we would have
\[ \left\lfloor \frac{n-1}{2} \right\rfloor = \sigma(P_2 \times P_{n-2}) \leq |D^n| = \left\lfloor \frac{n+1}{2} \right\rfloor - 2 = \left\lfloor \frac{n-3}{2} \right\rfloor < \left\lfloor \frac{n-1}{2} \right\rfloor, \]

a contradiction. Therefore, it must be the case that

\[ \sigma(P_2 \times P_{n+1}) \geq \left\lfloor \frac{n+2}{2} \right\rfloor \]

and the result follows. \( \square \)

**Theorem 8.** \( \sigma(P_3 \times P_n) = n - \left\lfloor \frac{n-1}{4} \right\rfloor. \)

**Proof.** Consider \( P_3 \times P_n \) as three canonical copies of \( P_n \) with vertices \( x_1, \ldots, x_n, y_1, \ldots, y_n \) and \( z_1, \ldots, z_n \) and with edges \( x_iy_i \) and \( y_iz_i \) between the copies for \( 1 \leq i \leq n \). To show \( (P_3 \times P_n) \leq n - \left\lfloor \frac{n-1}{4} \right\rfloor \): if \( n \) is odd, let \( D \) consist of those vertices \( x_i, z_i \) and \( y_j \), where \( i \equiv 3 \pmod{4} \) and \( j \equiv 1 \pmod{4} \). If \( n \) is even, let \( D \) consist of these same vertices together with \( y_n \). It is an easy matter to check that in each case \( D \) dominates \( P_3 \times P_n \) and that \( |D| = n - \left\lfloor \frac{n-1}{4} \right\rfloor. \)

As in the previous result, to show that \( \sigma(P_3 \times P_n) \geq n - \left\lfloor \frac{n-1}{4} \right\rfloor \) we proceed by induction. It is easy to see that

\[ \sigma(P_3 \times P_k) \geq k - \left\lfloor \frac{k-1}{4} \right\rfloor \]

for \( k = 1, 2, \ldots, 5 \). So, suppose for some \( k \geq 5 \)

\[ \sigma(P_3 \times P_n) \geq n_0 - \left\lfloor \frac{n_0-1}{4} \right\rfloor \]

for each \( n_0 \leq n \), and thus equality follows for each \( n_0 \). Furthermore suppose

\[ \sigma(P_3 \times P_{n+1}) < n + 1 - \left\lfloor \frac{n}{4} \right\rfloor. \]

Clearly this would imply that \( \sigma(P_3 \times P_{n+1}) = n - \left\lfloor \frac{n-1}{4} \right\rfloor \) since \( \sigma(P_3 \times P_n) = n - \left\lfloor \frac{n-1}{4} \right\rfloor \). Note for \( n - \left\lfloor \frac{n-1}{4} \right\rfloor < n + 1 - \left\lfloor \frac{n}{4} \right\rfloor \) it must be the case that \( n \equiv 0 \pmod{4} \). Let \( D \) be a minimum dominating set of \( (P_3 \times P_{n+1}) \) with \( n - \left\lfloor \frac{n-1}{4} \right\rfloor \) vertices. We consider various cases.

**Case 1.** Suppose \( D \) contains two or more of the vertices \( x_{n+1}, y_{n+1} \) and \( z_{n+1} \). This would imply that \( n - 1 - \left\lfloor \frac{n-2}{4} \right\rfloor = \sigma(P_3 \times P_{n-1}) \leq n - \left\lfloor \frac{n-1}{4} \right\rfloor - 2 \). But this implies that \( \left\lfloor \frac{n-1}{4} \right\rfloor + 1 \leq \left\lfloor \frac{n-2}{4} \right\rfloor \) which is a contradiction.
Case 2. Suppose $D$ contains none of the vertices $x_{n+1}, y_{n+1}$ and $z_{n+1}$. Since $D$ is a dominating set it must be the case that $x_n, y_n$ and $z_n$ are all in $D$. But this would imply that

$$n - 2 - \lfloor \frac{n-3}{4} \rfloor = \sigma(P_3 \times P_{n-2}) \leq n - \lfloor \frac{n-1}{4} \rfloor - 3.$$

This would give us that $1 + \lfloor \frac{n-1}{4} \rfloor \leq \lfloor \frac{n-3}{4} \rfloor$ an obvious contradiction.

Case 3. Exactly one of $x_{n+1}, y_{n+1}$ and $z_{n+1}$ are in $D$.

Subcase a. Suppose $x_{n+1} \in D$. Clearly, $z_n \in D$ since $z_{n+1}$ must be dominated by a vertex of $D$. If any of $y_n, x_n$ or $z_{n-1} \not\in D$, then $D - u_1$ would dominate $P_3 \times P_{n+1} - \{x_{n+1}, y_{n+1}, z_{n+1}\}$. This would imply that $\sigma(P_3 \times P_n) < n - \lfloor \frac{n-1}{4} \rfloor$ a contradiction. Thus none of $y_n, x_n$ or $z_{n-1}$ can be in $D$. If $y_{n-1} \in D$ then $D - x_n \cup \{z_{n+1}\}$ would be a minimum dominating set of $P_3 \times P_n$, but then we would have case 1. So $y_{n-1} \not\in D$. Also if $z_{n-1} \in D$ then $D - z_n \cup \{y_{n+1}\}$ would dominate the graph and again Case 1 results. Thus, since none of $x_{n-1}, y_{n-1}$, nor $z_{n-1} \in D$ it must be the case that $x_{n-2}$, and $y_{n-2} \in D$. But this implies that

$$n - 4 - \lfloor \frac{n-5}{4} \rfloor = \sigma(P_3 \times P_{n-4}) \leq n - \lfloor \frac{n-1}{4} \rfloor - 4.$$

This yields $\lfloor \frac{n-1}{4} \rfloor \leq \lfloor \frac{n-5}{4} \rfloor$ a contradiction.

Subcase b. Suppose $y_{n+1} \in D$. If either $x_n$ or $z_n \in D$ then $y_{n+1}$ could be replaced in $D$ by $z_{n+1}$ or $x_{n+1}$ respectively and thus subcase a would result. If $y_n \in D$, then, as in subcase a, $D - y_{n-1}$ would dominate $P_3 \times P_{n+1} - \{x_{n+1}, y_{n+1}, z_{n+1}\}$ which gives a contradiction. Hence neither $x_n, y_n$ nor $z_n \in D$ which implies that $x_{n-1}$ and $z_{n-1}$ are both in $D$. Consequently,

$$n - 3 - \lfloor \frac{n-4}{4} \rfloor = \sigma(P_3 \times P_{n-3}) \leq n - \lfloor \frac{n-1}{4} \rfloor - 3.$$

This implies $\lfloor \frac{n-1}{4} \rfloor \leq \lfloor \frac{n-4}{4} \rfloor$ which gives $n = 0 \mod 4$ a contradiction. With all cases exhausted, we claim $\sigma(P_3 \times P_{n+1}) \geq n + 1 - \lfloor \frac{n}{4} \rfloor$ and the Theorem follows. □

The solution for $P_4$ does not work out as nicely. We will prove the following:
Theorem 9. For all \( n \)

\[
\sigma(P_4 \times P_n) = \begin{cases} 
  n + 1 & \text{if } n = 1,2,3,4,5,6 \text{ or } 9 \\
  n & \text{otherwise.}
\end{cases}
\]

To prove Theorem 9, we present a number of Lemmas.

Lemma 10. For all \( n \), \( \sigma(P_4 \times P_n) \geq n \).

Proof. Suppose this were not the case. Let \( m \) be the smallest integer for which \( \sigma(P_4 \times P_m) < m \). Let \( D \) be a minimum dominating set for \( G = P_4 \times P_m \). Denote the columns of \( G \) by \( C_1, C_2, \ldots, C_m \) and note that at least one column does not intersect \( D \), since \( |D| < m \). Let \( j \) be the smallest integer for which \( C_j \cap D = \emptyset \). Since each vertex of \( C_j \) must be dominated by a vertex in \( C_{j-1} \) or in \( C_{j+1} \) we have

\[ |(C_{j-1} \cup C_{j+1}) \cap D| \geq 4. \]

Let \( F_1 \) be the vertices in \( C_1 \cup C_2 \cup \cdots \cup C_{j-1} \) and \( F_2 \) be the vertices in \( C_{j+1} \cup C_{j+2} \cup \cdots \cup C_m \). Note that \( F_1 \cap D \) dominates \( <F_1> \cong P_4 \times P_{j-1} \) and \( F_2 \cap D \) dominates \( <F_2> \cong P_4 \times P_{m-j} \). Since \( j - 1 < m \) and \( m - j < m \) it follows that \( |F_1 \cap D| \geq j - 1 \) and \( |F_2 \cap D| \geq m - j \), thus, \( m - 1 \geq |D| = |F_1 \cap D| + |F_2 \cap D| \geq (j-1) + (m+j) = m - 1 \). Hence, equality exists throughout and \( |F_1 \cap D| = j - 1 \) and \( |F_2 \cap D| = m - j \).

Now if \( j = 1 \), then \( F_1 \cap D = \emptyset \) and \( |C_2 \cap D| = 4 \). If \( j > 1 \), then \( F_1 \) has \( j - 1 \) columns, each intersecting \( D \) exactly once. So \( |C_{j+1} \cap D| \geq 3 \) and clearly then \( j \neq m \). If \( j = m - 1 \) then

\[ |D| = |F_1| + |C_m \cap D| \geq (m - 2) + 3 \]

which is a contradiction. If \( j = m - 2 \) then

\[ |D| = |F_1| + |C_{m-1} \cap D| + |C_m \cap C| \geq m - 3 + 3 \geq m \]

again a contradiction. Hence \( j \leq m - 3 \). Note that \( E = (C_{j+2} \cup \cdots \cup C_m) \cap D \) dominates \( <C_{j+3} \cup \cdots \cup C_m> \).

Since \( m - (j+2) < m \), we have

\[ |E| \geq \sigma(P_4 \times P_{m-j-2}) \geq m - j - 2. \]

Thus,

\[ |D| = |F_1| + |C_{j+1} \cap D| + |E| \geq (j-1) + 3 + m - j - 2 = m. \]

This is a contradiction, and the result follows. \( \square \)
Lemma 11. If \( n = 4, 7, 8 \) or if \( n \geq 10 \) then
\[
\sigma(P_4 \times P_n) \leq n.
\]

Proof. To show that \( \sigma(P_4 \times P_n) \leq n \) in these cases, we will display appropriate dominating sets. For convenience we label the vertices of the four disjoint \( P_n' \)’s:
\[
w_1, w_2, \ldots, w_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \text{ and } z_1, z_2, \ldots, z_n
\]
respectively. Also, let \( w_i, x_i, y_i, z_i \in E(G) \).

First, suppose \( n = 4 + 3k \) for some integer \( k \geq 0 \). Let \( D_1 = \{w_3, x_1, y_4, z_2\}, \quad D_2 = \{x_{6t+1}, x_{6t}, w_{6t}: t = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \) and \( D_3 = \{w_{6t+3}, y_{6t+4}, z_{6t+2}: t = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \). Set \( D = D_1 \cup D_2 \cup D_3 \). It is easy to show that \( D \) is a dominating set of \( P_4 \times P_n \) of order \( n \).

Now, suppose \( n = 8 + 3k \) for some integer \( k \geq 0 \). Let \( D_1 = \{w_3, x_1, y_4, z_2, w_7, x_5, y_8, z_6\}, \quad D_2 = \{x_{6t+4}, x_{6t+5}, w_{6t+3}: t = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \) and \( D_3 = \{w_{6t+7}, y_{6t+8}, x_{6t+6}: t = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \). Set \( D = D_1 \cup D_2 \cup D_3 \). Again, it can be shown that \( D \) is a dominating set of \( P_4 \times P_n \) of order \( n \).

Finally, suppose \( n = 12 + 3k \) for some integer \( k \geq 0 \). Let \( D_1 = \{w_3, w_7, x_{11}, x_3, x_5, y_4, y_8, y_{12}, z_2, x_6, x_{10}\}, \quad D_2 = \{x_{6t+8}, x_{6t+9}, w_{6t+7}: t = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \) and \( D_3 = \{w_{6t+11}, y_{6t+12}, x_{6t+10}: t = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \} \). Set \( D = D_1 \cup D_2 \cup D_3 \). As above, \( D \) is a dominating set of \( P_4 \times P_n \) of order \( n \). \( \square \)

These three constructions show that \( (P_4 \times P_n) \leq n \) for all \( n \) except \( 1, 2, 3, 5, 6 \) or \( 9 \).

We are now prepared to complete the proof of Theorem 9.

Proof of Theorem 9. Clearly, Lemmas 10 and 11 give \( (P_4 \times P_n) = n \) for \( n = 4, 7, 8 \) and \( n \geq 10 \). From Theorem A, Theorem 7 and Theorem 8 we get \( \sigma(P_4 \times P_n) = n + 1 \) for \( n = 1, 2 \) or \( 3 \). It only remains to show that \( \sigma(P_4 \times P_n) = n + 1 \) for \( n = 5, 6 \) and \( 9 \). For the remainder of this proof we will label the vertices of the four disjoint \( P_n \)’s as \( w_1, w_2, \ldots, w_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) and \( z_1, z_2, \ldots, z_n \).

First, we show that \( \sigma(P_4 \times P_5) = 6 \). Let \( D = \{w_1, w_5, x_3, y_3, z_1, z_5\} \). Clearly, \( D \) is a dominating set of \( P_4 \times P_5 \), and thus \( \sigma(P_4 \times P_5) \leq 6 \). Suppose \( D \) is a minimum dominating of \( P_4 \times P_5 \). If \( \{w_i \cap D = \emptyset \) it
must be the case that \(|\{z_i\} \cap D| = 5\). But this would imply that \(|D| > 5\). Hence, we may suppose that \(|\{w_i\} \cap D| \geq 1\), symmetrically \(|\{x_i\} \cap D| \geq 1\). Also, if \(|\{w_i\} \cap D| = |\{z_i\} \cap D| = 1\), this implies that \(|\{z_i\} \cap D|, |\{y_i\} \cap D| \geq 2\) which gives \(|D| \geq 6\). Thus, suppose, without loss of generality, that \(|\{w_i\} \cap D| \geq 2\). Also, it follows that one of \(|\{y_i\}| \cap D| or \(|\{z_i\} \cap D| \geq 2\). So, if \(|D| = 5\) one of \(|\{y_i\} \cap D| or \(|\{z_i\} \cap D| = \emptyset\), say \(|\{x_i\} \cap D| \geq 5\) which implies \(|D| \geq 6\). Hence \(\sigma(P_4 \times P_6) \geq 6\) and thus \((P_4 \times P_3) = 6\).

Next, consider \((P_4 \times P_6)\). Let \(D = \{w_2, w_3, z_4, y_1, y_5, x_3, x_6\}\). Clearly, \(D\) is a dominating set of \(P_4 \times P_6\), hence \(\sigma(P_4 \times P_6) \leq 7\). Let \(D\) be a minimum dominating set of \(P_4 \times P_6\). If \(|D| = 6\) then, since \(\sigma(P_4 \times P_3) = 4\) it must be the case that \(|\{w_1, w_2, x_1, w_2, y_2, y_2, z_1, z_2\} \cap D| \leq 2\). Symmetrically, \(|\{w_5, w_6, x_5, x_6, y_5, y_6, z_5, z_6\} \cap D| \leq 2\). Hence, we may suppose that \(D = D\) and thus \(D \notin D\). This implies that \(D \notin D\). But then \(|D| > 6\). Hence, \(x_6 \notin D\), which implies that \(x_6 \notin D\). Consequently, it follows that \(|D| > 6\) and thus \(\sigma(P_4 \times P_6) = 7\).

Finally, consider \(P_4 \times P_9\). Let \(D = \{w_1, w_2, w_3, x_4, z_9, y_1, y_7, z_3, z_5, z_6\}\). Clearly, \(D\) is a dominating set of \(P_4 \times P_9\), hence \(\sigma(P_4 \times P_9) \leq 10\). Let \(D\) be a minimum dominating set. If \(|D| = 9\), then since \(\sigma(P_4 \times P_6) = 7\) it follows that \(|\{w_1, w_2, x_1, w_2, y_2, y_2, w_2, z_2\} \cap D| \leq 2\). Symmetrically, \(|\{w_8, w_9, w_9, z_8, w_9, z_9, z_9, z_9\} \cap D| \leq 2\). Hence, we may assume \(y_1 \notin D\) thus \(z_2 \notin D\) and \(z_3 \notin D\). Since \(\sigma(P_4 \times P_9) = 6\), \(z_3 \notin D\), thus \(z_4 \notin D\). A symmetric argument shows that four vertices from \(\{w_i, x_i, y_i, z_i; i = 6, 7, 8, 9\}\) must be in \(D\), with \(z_6\) or \(y_6\) the one vertex from \(\{w_6, x_6, y_6, z_6\}\). Clearly, with these 8 vertices in \(D\) it is impossible for \(|D| = 9\). Thus \(\sigma(P_4 \times P_9) \geq 10\), and the theorem follows. \(\square\)

While Theorems 7, 8, and 9 might lead one to conjecture that, for all \(m\) and \(n\), \(\sigma(P_m \times P_n)\) is approximately \(\frac{mn}{4}\), this is not the case as can be seen by the following result. It also indicates the difficulty in finding \(\sigma(P_m \times P_n)\) for all \(m\) and \(n\).

**Theorem 12.** \(\lim_{m,n \to \infty} \frac{\sigma(P_m \times P_n)}{mn} = \frac{1}{5}\).
Proof. Label the vertices of $P_m \times P_n$ as $x_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Consider the vertices $x_{ij}$ where $j = 2i \pmod{5}$. There are no more than $\frac{mn}{5}$ of these vertices, and they dominate all of $P_m \times P_n$ except for approximately $\frac{1}{5}$ of those vertices of the form $x_{ij}, x_{mj}, x_{il}, x_{in}$. It is easier, and sufficient, to show that no more than $\frac{2}{5} (m + n) + 2$ vertices in $P_m \times P_n$ are not dominated by this set. Thus, using Lemma 1 for the lower inequality,

$$\frac{1}{5} \leq \frac{\sigma(P_m \times P_n)}{mn} \leq \frac{1}{mn} \left( \frac{mn}{5} + \frac{2}{5} (m + n) + 2 \right).$$

Since the expression on the right approaches $\frac{1}{5}$ for large $m$ and $n$, the result follows. \[ \Box \]

Conclusions and Questions. There are a great deal of tractable questions related to this conjecture. One posed in [3], is to characterize the graphs $G$ and $H$ for which $\sigma(G \times H) = \sigma(G) \cdot \sigma(H)$. The obvious question here is to determine for all $m$ and $n$, $\sigma(P_m \times P_n)$. In [6], bounds are determined for $\sigma(P_m \times C_n)$, and thus for $\sigma(P_m \times P_n)$, by transforming the problem into an integer programming problem. Finally, there is a problem of determining $\sigma(G \times H)$ precisely for other classes of graphs.
References.


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