Fragile Graphs with Small Independent Cuts

Guantao Chen,1* Ralph J. Faudree,2
and Michael S. Jacobson3

1DEPARTMENT OF MATHEMATICS AND STATISTICS
GEORGIA STATE UNIVERSITY
ATLANTA, GEORGIA 30303
E-mail: gchen@cs.gsu.edu

2DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF MEMPHIS
MEMPHIS, TENNESSEE 38152

3DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KENTUCKY 40292

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Abstract: A graph is called fragile if it has a vertex cut which is also an
independent set. Chen and Yu proved that every graph with \( n \) vertices and
at most \( 2n - 4 \) edges is fragile, which was conjectured to be true by Caro.
However, their proof does not give any information on the number of
vertices in the independent cuts. The purpose of this paper is to investigate
when a graph has a small independent cut. We show that if \( G \) is a graph on

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*Correspondence to: Guantao Chen, Department of Mathematics and Statistics,
Georgia State University, Atlanta, GA 30303-3083. E-mail: gchen@cs.gsu.edu

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If a graph $G$ has $n$ vertices and at most $(12n/7) - 3$ edges, then $G$ contains an independent cut $S$ with $|S| \leq 3$. Upper bounds on the number of edges of a graph having an independent cut of size 1 or 2 are also obtained. We also show that for any positive integer $k$, there is a positive number $\varepsilon$ such that there are infinitely many graphs $G$ with $n$ vertices and at most $(2 - \varepsilon)n$ edges, but $G$ has no independent cut with less than $k$ vertices.

Keywords: independent set; cut; fragile graphs

1. INTRODUCTION

We will only consider finite graphs without loops or multiple edges. Given a graph $G$, the vertex set of $G$ and the edge set of $G$ will be denoted by $V(G)$ and $E(G)$, respectively. Let $|G| = |V(G)|$ and $|E(G)| = |E(G)|$. For a connected graph $G$, a subset $S \subseteq V(G)$ is called a cut of $G$ if $G - S$ is no longer connected. A vertex set $S \subseteq V(G)$ is independent if no two vertices of $S$ are joined by an edge. A cut $S$ of $G$ which is also an independent set is called an independent cut. If $G$ is disconnected, for convenience, we define $\emptyset$ as an independent cut of $G$. A simple connected graph on at least three vertices is called a fragile graph if it contains an independent cut. Caro and Yuster [4] noted that fragile graphs play a role in some decomposition algorithms. deFigueiredo and Klein [6] proved that deciding if a graph is fragile is NP-complete. Clearly, every triangle-free connected graph with at least three vertices is fragile. Brandstädte, Drangan, Van B. Le, and Szyczack proved that deciding if a $K_4$-free graph is fragile is also NP-complete. It is naturally expected that sparse graphs (graphs with few edges) are fragile. However, the graph obtained from $K_{2,n-2}$ by adding an edge joining the two vertices of the part with just two vertices is not fragile. This graph has $n$ vertices and $2n - 3$ edges. On the other hand, the following result was conjectured to be true by Caro [3] and proved by Chen and Yu [5].

**Theorem 1.1.** If $|G| \leq 2|G| - 4$, then $G$ is fragile.

Since Theorem 1.1 was proved by induction, it does not give an indication of the size of the independent cuts. Caro [3] asked if the same condition implies that the graph has an independent cut of size at most three. We will show that the answer to Caro’s question is negative. In Section 2, we will prove that for each positive integer $k$, there exists a positive number $\varepsilon$ such that there are infinitely many graphs $G$ with $|G| \leq (2 - \varepsilon)|G|$ and no independent cut of less than $k$ vertices. On the other hand, we obtain an upper bound on the number of edges that implies a graph contains an independent cut of size at most 3. Also, this bound is shown to be sharp.
For any graph $G$, let $f(G)$ be the minimum size of independent cut of $G$, if $G$ contains an independent cut. Otherwise, let $f(G) = \infty$. For any pair of positive integers $n$ and $m$ with $m \leq 2n - 4$, we define:

$$f_n(m) = \max\{f(G) : G \text{ is a graph with } n \text{ vertices and } m \text{ edges}\}$$

By Theorem 1.1, $f_n(m)$ is finite. Clearly, if $m_1 < m_2 \leq 2n - 4$, then $f_n(m_1) \leq f_n(m_2)$.

For positive integers $k$ and $n$, our subject is to determine the maximum $m$ such that $f_n(m) = k$. Since every graph with $n$ vertices and $n - 1$ edges is either disconnected or a tree, then $f_n(n - 1) = 1$. Let $C_n$ be any cycle on $n$ vertices. Since the connectivity of $C_n$ is 2, we have $f_n(n) \geq 2$.

**Theorem 1.2.** For any positive integer $n$, if $m < \min\{2n - 3, 3n/2\}$, then $f_n(m) \leq 2$.

**Proof.** The result is vacuous for $n \leq 2$, since $2n - 4 \leq 0$ for $n \leq 2$. The only graph with three vertices and two edges is a path of length 2. The result is true for $n \leq 3$. Suppose that $n \geq 4$. Let $G$ be a graph with $|G| = n$ and $|G| = m < \max\{2n - 3, 3n/2\}$. Clearly, $G$ has an independent cut of size at most 1 if $G$ either is disconnected or has a vertex of degree 1. Assume minimum degree $\delta(G) \geq 2$. Since $|G| < 3n/2$, $\delta(G) = 2$. Let $v \in V(G)$ such that $d(v) = 2$ and let $x$ and $y$ be the neighbors of $v$. If $x$ and $y$ are not adjacent, $\{x, y\}$ is an independent cut of $G$. Assume that $xy \in E(G)$. Let $H = G - v$. Then, $H$ has $n - 1$ vertices and $m - 2$ edges. By induction hypothesis, $H$ contains an independent cut $T$ of size at most 2. Clearly, $T$ is also an independent cut of $G$, since $xy \in E(G)$.

Note $3n/2 \leq 2n - 3$ for $n \geq 6$. Any cubic 3-connected graph with $n$ vertices has exactly $3n/2$ edges. Thus, Theorem 1.2 is best possible.

For each positive integer $t \geq 2$, we will construct a graph $H$ with $7t$ vertices and $12t$ edges, but no independent cut of at most three vertices.

Let $C = v_1v_2 \ldots v_{2t}v_1$ be a cycle with $2t$ vertices. For each $i = 1, 2, \ldots, t$, set $A_i = \{v_i, v_{i+1}, v_{i+t}, v_{i+t+1}\}$ and add a bowtie $B_i$ with vertex set $\{x_i, y_i, z_i, y_{t+i}, z_{t+i}\}$ and edge set $\{x_iy_i, x_iz_i, x_iy_{t+i}, x_iz_{t+i}, y_iz_i, y_{t+i}z_{t+i}\}$. Bowtie $B_i$ and $C$ are joined by the edges $y_iv_i, z_iv_{i+1}, y_{t+i}v_{t+i}, z_{t+i}v_{t+i+1}$. Let $H_t$ denote the resulting graph. Clearly, $|H_t| = 7t$ and $|H_t| = 12t$. Further, it is readily seen that $H_t$ is 3-connected and every independent cut of three vertices must have exactly two vertices on the cycle $C$. Using this fact, we can show that $H_t$ does not contain any independent cut of three vertices.

On the other hand, we will show that the following result holds.
Theorem 1.3. For any positive integer \( n \), \( f_n(12n/7 - 3) \leq 3 \). Thus, every graph \( G \) with \( n \) vertices and at most \( (12n/7 - 3) \) edges contains an independent cut of size at most 3.

In the next section, we will construct examples which do not contain a small independent cut.

2. AN EXAMPLE

For any positive integer \( k \), we will show that there is a positive number \( \varepsilon > 0 \) such that there are infinitely many graphs \( G \) with \( ||G|| \leq (2 - \varepsilon)||G| \) and \( f(G) \geq k \).

Examples are constructed based on the following result of Bollobás [1] and Kostocka and Mazurova [7].

Theorem 2.1. Let \( k \geq 4 \), \( c = \lfloor 4k(1 + \log k) \rfloor \) and \( \Delta_0 = \Delta(k) = \lfloor 4ec \rfloor \), where \( e \approx 2.78 \ldots \) is the natural number. Then for every sufficiently large \( n \), there is a graph \( G \) of order \( n \) such that \( \Delta(G) \leq \Delta_0 \), \( \chi(G) \geq k \) and \( g(G) \geq g_0 \), where \( g_0 \) is the maximal integer satisfying

\[
\frac{1}{g_0}(2c)^{g_0} < \frac{n}{12k^2}.
\]

A graph \( G \) is called critical if \( \chi(G - v) < \chi(G) \) for every \( v \in V(G) \). It is an easily derived fact that every critical graph has edge-connectivity \( \lambda(G) \geq \chi(G) - 1 \). Thus, every graph with \( \chi(G) \geq k + 1 \) has a \( k \) edge-connected subgraph. Applying Theorem 2.1 again and again with the girth \( g \) increasing to infinity, we obtain the following result.

Theorem 2.2. For every positive integer \( k \geq 2 \) there are infinitely many \( k \)-edge-connected graphs \( H \) with \( \Delta(H) \leq \Delta = \lfloor 4e[4(k + 1)(1 + \log(k + 1))] \rfloor \) and girth \( g(H) \geq k \).

For each graph \( H \) satisfying Theorem 2.2, let \( S \) be a maximum vertex set of \( H \) such that

\[
dist(u, v) \geq k \quad \text{for any pair vertices } u \text{ and } v \in S.
\]

Since the maximum degree \( \Delta(H) \leq \Delta \), there is an

\[
|S| \geq \frac{|H|}{\Delta(\Delta - 1)^{k-2} + 1}.
\]

Let \( \varepsilon_0 = 1/(\Delta(\Delta - 1)^{k-2} + 1) \). Clearly, \( \varepsilon \) only dependents on \( k \). From a new graph \( H^* \) from \( H \) with the following two step construction.

1. Replace each vertex \( v \in V(H) \) by a cycle \( C_{d(v)} \), where \( d(v) \) is the degree of \( v \) in \( H \). The cycle \( C_{d(v)} \) is called a hoop cycle. Each vertex of the hoop
\(C_d(v)\) is incident to one of the edges at \(v\), and delete \(v\). The resulting graph \(H'\) is 3-regular.

(2) For each vertex \(v \in S\), arbitrarily choose two consecutive vertices \(v_1\) and \(v_2\) on the hoop cycle \(C_d(v)\) and delete the edge \(v_1v_2\).

Clearly, \(|H^*| = 2||H|| \leq \Delta|H|\) and all vertices of \(H^*\) have degree 3 except for \(2|S| \geq 2|H|\varepsilon_0\) vertices of degree 2.

Given a graph \(G\), an edge set \(T \subseteq E(G)\) is called a cycle edge-cut if \(T\) is an edge cut of \(G\) and each component of \(G - T\) contains a cycle. The cycle edge-connectivity of a graph \(G\) is the least \(|T|\), where \(T\) is a cycle edge-cut.

**Claim 2.1.** \(H^*\) has cycle edge-connectivity at least \(k\).

**Proof.** Assume, to the contrary, \(H^*\) has a cycle edge-cut \(T\) with \(|T| \leq k - 1\). Without loss of generality, we can assume that \(H^* - T\) has exactly two components \(A\) and \(B\). If both \(A\) and \(B\) contain all vertices of a hoop cycle, respectively, then \(T \cap E(H)\) is an edge cut of \(H\), which contradicts the assumption that \(H\) is \(k\) edge-connected.

Without loss of generality, assume that \(A\) does not contain all vertices of any hoop cycle. Let \(C_1\) be a cycle in \(A\). Since girth \(g(H) \geq k\), \(C_1\) intersects with at least \(k\) hoop cycles. Since \(A\) does not contain all vertices of any hoop cycle, each hoop cycle intersected with \(A\) contributes at least one edge to \(T\). Thus, \(|T| \geq k\). \[\Box\]

Let \(L = L(H^*)\) be the line graph of \(H^*\). Since all vertices of \(H^*\) have degree 3 except those \(2|S|\) vertices on hoop cycles induced by vertices in \(S\), \(|L| = (3/2)|H^*| - |S| \leq (3/2)|H^*|\). Since \(S\) is, in particular, an independent set of \(H\) and \(H\) is 3-regular, all original edges of \(H\) (not in \(S\)) have degree at least 3 in \(L\). Thus, \(L\) has all vertices of degree 4, except \(4|S|\) vertices of degree 3. Note that

\[
4|S| \geq 4|H|\varepsilon_0 \geq \frac{4\varepsilon_0|H^*|}{\Delta} \geq \frac{(8/3)\varepsilon_0|L|}{\Delta}.
\]

Let \(\varepsilon = \frac{(4/3)\varepsilon_0}{\Delta}\). We obtained that,

\[
||L|| \leq (2 - \varepsilon)|L|.
\]

We claim that \(f(L) \geq k\). Suppose, to the contrary, let \(F\) be an independent vertex cut of \(L\) with \(|F| \leq k - 1\). Let \(F^*\) be the corresponding edge set of \(F\) in \(H^*\). Then, \(F^*\) is an independent edge-cut of \(H^*\). Since the cycle edge-connectivity of \(H^*\) is at least \(k\), there is a component \(A\) of \(H^* - F^*\) which is a tree. Further, we can assume that \(A\) does not contain all vertices of any hoop cycle. Since \(F^*\) is an independent edge-cut, \(|A| \geq 2\). Thus, \(A\) contains two vertices, say \(u\) and \(v\), of degree 2 in \(H^*\). Since \(A\) does not contain all vertices of a hoop cycle, the path connecting \(u\) and \(v\) in \(A\) must intersect with \(k\) the vertices in \(k\) different hoop
cycles. Since each hoop cycle contributes at least one edge in \(F^*\), \(|F^*| \geq k\), a contradiction.

Then, we have proved the following theorem.

**Theorem 2.3.** For each positive integer \(k\), there exists an \(\varepsilon\) such that there are infinite many graphs \(L\) with \(|L| \leq (2 - \varepsilon)|L_0|\) and \(L\) does not contain an independent cut with less than \(k\) vertices. Further, \(\varepsilon \to 0\) as \(k \to \infty\).

## 3. PROOF OF THEOREM 1.3

The proof is by way of induction on \(|G|\). The result is clearly true for \(|G| \leq 3\). Suppose, \(|G| = n \geq 4\) and the result is true for graphs with order less than \(n\).

Assume, to the contrary, every independent cut of \(G\) has at least four vertices. We will first make some observations that will be used in the proof of Theorem 1.3.

### A. \(G\) Is 3-Connected

**Proof.** Clearly, \(G\) must be 2-connected, otherwise, \(G\) will contain an independent cut of either 0 or 1 vertex. Now suppose that \(G\) is not 3-connected. Let \(S = \{u, v\}\) be a cut of \(G\). Since \(S\) is not independent, it follows that \(uv \in E(G)\). Let \(G_1\) and \(G_2\) be two induced subgraphs of \(G\) such that \(G = G_1 \cup G_2\) and \(V(G_1) \cap V(G_2) = S\). By the induction hypothesis, we have \(|G_1| \geq \frac{12}{7}|G_1| - 3\) and \(|G_2| \geq \frac{12}{7}|G_2| - 3\), since an independent 3 cut in \(G_1\) or \(G_2\) would also be independent 3 cut in \(G\). Thus,

\[
\left(\frac{12}{7}\right)n - 3 \geq ||G|| = ||G_1|| + ||G_2|| - 1 \geq \frac{12}{7}|G_1| - 3 + \frac{12}{7}|G_2| - 3 + 1
\]

\[
= \frac{12}{7}(|G| + 2) - 5 > \frac{12}{7}n - 3,
\]

a contradiction. \(\blacksquare\)

As a consequence of (A), clearly, we may assume that \(\delta(G) \geq 3\). Let \(V_3\) be the set of vertices of degree 3 in \(G\).

### B. \(G[V_3]\) Does Not Contain a \(K_3\)

**Proof.** Suppose, to the contrary, \(G[V_3]\) contains a \(K_3\) with vertices \(x, y,\) and \(z\). Let \(x^*, y^*,\) and \(z^*\) be three other vertices such that \(xx^*, yy^*,\) and \(zz^* \in E(G)\). Since \(G\) is 3-connected, \(x^*, y^*,\) and \(z^*\) must be three distinct vertices. If \(x^*y^* \notin E(G)\), then \(\{x^*, y^*, z^*\}\) is an independent cut of three vertices, a contradiction. thus, \(x^*y^* \in E(G)\). Similarly, \(y^*z^* \in E(G)\) and \(x^*z^* \in E(G)\). Hence, \(G[\{x^*, y^*, z^*\}] = \{x, y, z\}\). Thus, \(H = G - \{x, y, z\}\) has \(n - 3\) vertices and at most \((12n/7) - 3 - 6 < (12(n - 3)/7) - 3\) edges. By our induction hypothesis, \(H\) has an independent cut.
S of no more than three vertices. Clearly, S is also an independent cut of G, a contradiction.

C. For Any \( v \in V(G) \), \( G[N(v)] \) Is a Disconnected Subgraph

Proof. Suppose, to the contrary, \( G[N(v)] \) is connected. Then, it contains a spanning tree \( T_v \). Let u be a leaf of the spanning tree. Let \( H = (G - v) \cup \{wy : y \in N(v)\} \). Clearly, \( |H| = |G| - 1 \) and \( ||H|| \leq ||G|| - d(v) + (d(v) - 2) = ||G|| - 2 \). By our induction hypothesis, H has an independent cut S with \( |S| \leq 3 \). Since S is independent, S is also a cut of G, a contradiction.

Since every independent cut of G contains at least four vertices, the subgraph induced by neighbors of a vertex of degree three contains exactly one edge.

D. \( G[V_3] \) Does Not Contain a \( P_4 \)

Proof. Suppose, to the contrary, \( G[V_3] \) contains a path \( v_1v_2v_3v_4 \). Let \( w_2 \in N(v_2) - \{v_1, v_3\} \). We have either \( w_2v_1 \in E(G) \) or \( w_2v_3 \in E(G) \). If \( w_2v_3 \in E(G) \), then \( \{v_1, w_2, v_4\} \) is a cut. By (C), we have \( w_2v_1 \notin E(G) \) and \( w_2v_4 \notin E(G) \). Thus, \( v_1v_4 \in E(G) \). Since \( G[V_3] \) does not contain a triangle, \( v_1v_3 \notin E(G) \) and \( v_4v_2 \notin E(G) \). Let \( u \in N(v_1) - \{v_2, v_4\} \). Since \( N(v_2) = \{v_1, v_3, w_2\} \), \( \{v_2, v_4\} \notin E(G) \). Thus, \( uv_4 \in E(G) \), which implies that \( \{u, w_2\} \) is a cut of two vertices, a contradiction. Therefore, \( w_2v_1 \in E(G) \). Let \( w_3 \in N(v_3) - \{v_2, v_4\} \). Similarly, we can show that \( w_3v_4 \in E(G) \).

Let \( w_1 \in N(v_1) - \{w_2, v_2\} \) and \( w_4 \in N(v_4) - \{w_3, v_3\} \). Since \( \{w_1, w_2, v_3\} \) is a cut of G, then \( w_1v_3 \in E(G) \). Since \( d(v_3) = 3 \), we have that \( w_3 = w_1 \). Similarly, \( w_2 = w_4 \). Thus \( \{w_2, w_4\} \) is a cut of G, a contradiction to G being 3-connected.

E. Let u and v Be Two Vertices in \( V_3 \) With uv \( \in E(G) \). If Edge uv Is on a Triangle uvwvu, Then There Are Two Vertices \( u^* \) and \( v^* \) Such That \( \{u, v, u^*, v^*\} \) Induces a \( C_4 \)

Proof. Let \( u^* \in N(u) - \{v, w\} \) and \( v^* \in N(v) - \{u, w\} \). Since \( G[N(u)] \) is not connected, \( u^* \neq v^* \). Since \( G[N(u)] \) contains exactly one edge, \( u^*w \notin E(G) \). For the same reason, \( v^*w \notin E(G) \). Since \( \{u^*, v^*, w\} \) is a cut, and not an independent cut, \( u^*v^* \in E(G) \). Thus, \( uvu^*u \) is an induced cycle.
If $G[V_3]$ contains a path $xyz$, then there are three other vertices $x^*, y^*$, and $z^*$ with $xy^*$ and $yz^* \in E(G)$ and either $G[\{x, x^*, y^*\}]$ and $G[\{y, z, z^*\}]$ are triangles in $G$ or $G[\{x, y, x^*\}]$ and $G[\{z, y^*, z^*\}]$ are triangles in $G$, and $d(y^*) \geq 5$. We call $y^*$ the fat vertex of the path $xyz$. Note that these two cases are symmetric regarding to the path $xyz$. Further, if $y^*$ is the fat vertex of $k$ disjoint 3 paths in $G[V_3]$, then $d(y^*) \geq 4 + k$

\[
\begin{array}{c}
3 \quad 3 \quad 3 \\
\end{array}
\quad \Rightarrow
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\end{array}
\quad >5
\]

\textbf{Proof.} Let $z^* \in N(y) - \{x, z\}$. Since $N(y)$ is not independent, without loss of generality, we may assume that $z^*z \in E(G)$. By (E), there is a vertex $y^*$ such that $xy^*zxy$ is an induced 4-cycle. Let $x^*$ be the other neighbor of $x$ besides $y$ and $y^*$. Since $x^*y \not\in E(G)$ and $yy^* \not\in E(G)$, we obtain that $x^*y^* \in E(G)$. Thus, $x^*y^*zxy$ is a $K_4$. We show that $d(y^*) \geq 5$.

Since $G[V_3]$ does not contain a $P_4$, $d(y^*) \geq 4$. Let $y^{**} \in N(y^*) - \{x, x^*, z\}$. If $d(y^{**}) = 4$, then $\{x^*, y^{**}, z^*\}$ is a cut of $G$. Then, $G[\{x^*, y^{**}, z^*\}]$ contains a least one edge. Let $G_1 = (G - \{x, y, z, y^*\}) \cup \{x^*y^{**}, y^{**}z^*, x^*z^*\}$. We have that,

\[
|G_1| = |G| - 4, \quad ||G_1|| \leq |G| - 9 + 2 = |G| - 7 \leq \frac{12}{7} |G| - 3 - 7 < \frac{12}{7} |G_1| - 3.
\]

By our induction hypothesis, $G_1$ has an independent cut $S$ of at most three vertices. Clearly, $S$ is also a cut of $G$, a contradiction.

Suppose that $y^*$ is the fat vertex of $k$ vertex disjoint 3 paths in $G[V_3]$. Since $G[V_3]$ contains no $P_4$, $d(y^*) \geq 2k + 1$. Since $2k + 1 \geq k + 4$ for $k \geq 3$, we only need to deal with the case, when $k = 2$. Let $y^*$ be the fat vertex for paths $xyz$ and $x_1y_1z_1$ in $G[V_3]$. We will show that $d(y^*) \geq 6$. Suppose, to the contrary, $d(y^*) = 5$. By the definition of fat vertex, we have that $y^*$ is adjacent to $x, y, x_1, y_1$. Without loss of generality, we assume that $xx^*y^*x$ is a triangle and $yz^*z$ is a triangle. Since $x^* \not\in V_3$ and $d(y^*) = 5$, either $x^*y^*x_1x^*$ is a triangle or $x^*y^*z_1z^*$ is a triangle. Without loss of generality, we assume that $x^*y^*x_1x^*$ is a triangle and assume that $z_1^*y_1z_1^*$ is another triangle involving with $x_1y_1z_1$. Then, $\{x^*, z^*, z_1^*\}$ is a cut of $G$. Then, $G[\{x^*, z^*, z_1^*\}]$ contains at least one edge. Let

\[
H = (G - \{x, y, z, x_1, y_1, z_1, y^*\}) \cup \{x^*, z^*, z_1^*\}.
\]

Then, $|H| = |G| - 7$ and $||H|| = |G| - 13 < (12/7)|H| - 3$. By the induction hypothesis, there is an independent cut $S$ of $H$ with $|S| \leq 3$. Clearly, $S$ is also a cut of $G$, a contradiction.
G. \( G \) Does Not Contain the Following Graph \( H_1, H_2, H_3, \) or \( H_4 \) as Induced Subgraphs. Where \( V(H_1) = \{x, y, u_1, v_1, u_2, v_2, u_3, v_3\} \) Such That \( u_i \neq v_i \) for Each \( i = 1, 2, 3. \) Further \( d(x) = 4, d(y) = 4, d(u_i) = d(v_i) = 3 \) for Each \( i = 1, 2, \) and 3. \( H_2 = H_1/uv \) the Graph Obtained From \( H_1 \) by Contracting the Edge \( uv \) to a new Vertex \( w \), \( H_3 = H_2/uv \) the Graph Obtained From \( H_2 \) by Contracting the Edge \( uv \) to a New Vertex \( w \), and \( H_4 = H_3/uv \) the Graph Obtained From \( H_3 \) by Contracting the Edge \( uv \) to a New Vertex \( w \)

![Graphs H1, H2, H3, H4](image)

**Proof.** Suppose, to the contrary, \( G \) has \( H_i \) as an induced subgraph. If \( i = 4 \), then \( \{w_1, w_2, w_3\} \) is an independent cut of three vertices, a contradiction.

If \( i = 3 \), using the fact that either \( G[N(u_3)] \) or \( G[N(v_3)] \) contains a triangle, \( N(u_3) \cap N(v_3) \neq \emptyset \). Let \( w \) be the common neighbor of \( u_3 \) and \( v_3 \), then \( \{w_1, w_2, w\} \) is an independent cut of three vertices, a contradiction.

If \( i = 2 \), let \( w_1 \in N(u_1) - \{x, y\} \) and \( w_i \) be the common neighbor of \( u_i \) and \( v_i \) for \( i = 2, 3 \). Then \( \{w_1, w_2, w_3\} \) is a cut of \( G \). Thus, \( G[w_1w_2w_3] \) contains at least one edge. Let \( G_1 = (G - V(H_2)) \cup \{w_1w_2, w_2w_3, w_3w_1\} \). Clearly, \( |G_1| = |G| - 7 \) and \( ||G_1|| = ||G|| - 13 < (12/7)||G|| - 3 \). By the induction hypothesis, \( G_1 \) has an independent cut \( S \) with \( |S| < 3 \), so does \( G \), a contradiction.

Assume that \( i = 1 \). Since every vertex of degree 3 must be in a triangle, let \( w_1, w_2, \) and \( w_3 \in G - V(H_1) \) such that \( w_1u_i \in E(G) \) and \( w_iv_i \in E(G) \). Since \( G \) is 3-connected, \( w_1, w_2, \) and \( w_3 \) are three distinct vertices. Since \( \{w_1, w_2, w_3\} \) is a cut, \( G[w_1w_2w_3] \) contains an edge. Let \( G_1 = (G - V(H_1)) \cup \{w_1w_2, w_2w_3, w_3w_1\} \). Clearly, \( |G_1| = |G| - 8 \) and \( ||G_1|| \leq ||G|| - 14 \leq (12/7)||G|| - 3 \). By the induction hypothesis, \( G_1 \) has an independent cut \( S \) with \( |S| \leq 3 \), so does \( G \), a contradiction.

From the results above, we see that each component of \( G[V_3] \) is either an isolated vertex, an edge, or a path with three vertices. We will define some special edges in the neighbors of \( V_3 \) according to the component structure as follows.
If $x$ is an isolated vertex in $G[V_3]$ and $xuwx$ is a triangle containing $x$, then $uv$ is called the primitive edge (type I) of $x$ and denoted by $p(x)$ and edges $xu$ and $xw$ are called supporting edges. Note that $p(x) = p(y)$ may hold even if $x \neq y$.

If edge $xy$ is a component of $G[V_3]$ and it is not on any triangle. Let $u_1$ and $v_1$ be the other two neighbors of $x$ and $u_2$ and $v_2$ be the other two neighbors of $y$. Since $xy$ is not on any triangle, we have that $u_1y \not\in E(G)$ and $v_1y \not\in E(G)$. Thus, $u_1v_1 \in E(G)$. Similarly, $u_2v_2 \in E(G)$. We name $u_1v_1$ the primitive edge (type I) of $x$ and $u_2v_2$ the primitive edge (type I) of $y$ and edges $xu_1$, $xv_1$, $yu_2$, and $yv_2$ are called supporting edges. In this case, we call vertices $x$ and $y$ semi-isolated vertices in $G[V_3]$.

If Edge $xy$ is a component of $G[V_3]$ and it is on a triangle, say $xywx$. By (E), there are two distinct vertices $x^*$ and $y^*$ such that $xxy^*x$ is an induced $C_4$. In this case, we call $x^*y^*$ a primitive edge (type II) of $xy$ and denote it by $p(xy)$. The vertex $w$ is called the clipper of $xy$ and is denoted by $c(xy)$. Edges $xx^*$, $yy^*$, $xw$, and $yw$ are called supporting edges. In this case, we call the edge $xy$ a type II edge.

If $xyz$ is a component of $G[V_3]$, by (F), we assume that, $u$, $v$, and $w$ are three neighbors of $\{x, y, z\}$ such that $uwzu$ and $wyzw$ are two triangles and $v_3 \in E(G)$ with $d(v) \geq 5$. In this case, we call $uv$ a primitive edge (type III) of $xyz$ and $w$ the clipper of $xyz$. Denote them by $p(xyz)$ and $c(xyz)$, respectively. Edges $xu$, $uv$, $yw$, $zw$, and $vz$ are called supporting edges.

We construct a graph $H$ from $G$ by splitting clippers and assign a weight $\omega$ to every vertex of $H$ as follows.

1. Initially, we define that $\omega(x) = 1$ for all vertices $x \in V(G)$.
2. If $c$ is a clipper of edge $xy$ (or path $xyz$), we split vertex $c$ into two vertices $c'$ and $c^*$ such that $c^*$ is adjacent to $x$ and $y$ ($c^*$ is adjacent to two original neighbors of $c$ in $\{x, y, z\}$), while $c'$ is adjacent to all other neighbors of $c$. Define $\omega(c') = \max(\omega(c) - 1/2, 0)$ and $\omega(c^*) = 1/2$. We call $c^*$ the company of $xy$ ($xyz$) and $c'$ the descendant of $c$.

For example, if $c$ is the clipper for two edges $xy$ and $uv$ with $\omega(c) = 1$, we first randomly pick one edge, say $xy$ and treat $c$ only as a clipper of edge $xy$ and split $c$ into the company $c^*$ and the descendant $c'$, i.e., $c^*$ is adjacent to both $x$ and $y$, while $c'$ is adjacent to the other neighbors of $c$ with $\omega(c^*) = 1/2$ and $\omega(c') = 1/2$. Then, we treat $c'$ as a clipper of $uv$ and splitting $c'$ into the company $(c')^*$ and the descendant $(c')'$. As the result, we have $\omega(c^*) = \omega((c')^*) = 1/2$ and $\omega((c')') = 0$.

For any $U \subseteq V(H)$, we let $\omega(U) = \sum_{v \in U} \omega(v)$. For any subgraph $F$ of $H$, we define $\omega(F) = \omega(V(F))$. Clearly,

$$\omega(H) = |G|$$

$$\sum_{u \in V(H)} d_H(u) = \sum_{v \in V(G)} d_G(v) \leq 2((12/7)|G| - 3) = (24/7)\omega(H) - 6.$$
Our proof strategy is to partition \( V(H) \) into \( Y_0, Y_1, \ldots, Y_m \) such that

\[
\sum_{v \in Y_i} d_H(v) \geq \frac{24}{7} \omega(Y_i) \quad \text{for each } i = 0, 1, \ldots, m,
\]

which gives a contradiction. The above inequality will be trivial if \( \omega(Y_i) = 0 \). Without loss of generality, we assume that \( \omega(Y_i) > 0 \) whenever we need the above inequality. In this sense, the above inequality is equivalent to the follows.

\[
\sum_{v \in Y_i} \frac{d_H(v)}{\omega(Y_i)} \geq \frac{24}{7} \quad \text{for each } i = 0, 1, \ldots, m.
\]

Clearly, if \( Y_i \) does not contain any vertex of degree \( \leq 3 \) in \( H \), then \( (\sum_{v \in V_i} d_H(v)) / \omega(V_i) \geq 4 \geq 24/7 \). Note that vertices of degree \( \leq 3 \) in \( H \) are those in \( V_3 \) plus their companies and descendants. Let \( X_1, X_2, \ldots, X_m \) be the vertex sets of components of the subgraph of \( G \) induced by all primitive edges. For each \( X_i \), let \( Y_i \) be a vertex set of \( H \) containing \( X_i \) and those components of \( G[V_3] \) whose primitives are in \( G[X_i] \) and their companies. Let \( Y_0 = V(H) - \bigcup_{i=1}^m Y_i \). Clearly, \( Y_0, Y_1, Y_2, \ldots, Y_m \) are disjoint subsets of \( V(H) \) and \( V_3 \subseteq \bigcup_{i=1}^m Y_i \).

For each \( v \in Y_0 \), if \( v \in V(G) \), we have \( d_H(v) = d_G(v) \geq 4 \). Then, \( d_H(v) / \omega(v) \geq 4 \). If \( v \in V(H) - V(G) \), we have either \( \omega(v) = 0 \) or \( 1/2 \). If \( \omega(v) = 1/2 \), \( d_H(v) \geq 4 - 2 = 2 \). Then, \( d_H(v) / \omega(v) \geq 4 \). Thus, in either case, we have \( d_H(v) / \omega(v) \geq 4 \). Therefore,

\[
\frac{\sum_{v \in Y_0} d_H(v)}{\omega(Y_0)} \geq 4 \geq \frac{24}{7}.
\]

Let \( F_i \) be a subgraph of \( H \) with vertex set \( Y_i \) and edge set consisting all supporting edges incident with \( Y_i \) and edges in \( G[X_i] \) for each \( i = 1, 2, \ldots, m \). Suppose that \( F_i \) contains \( t_1 \) isolated and semi-isolated vertices in \( V_3 \), \( t_2 \) type II edges of \( G[V_3] \), and \( t_3 \) \( P_3 \)'s in \( G[V_3] \). Let \( \gamma(F_i) = (\sum_{v \in Y_i} d_H(v)) / \omega(T) \). Our goal is to show that \( \gamma(F_i) \geq 24/7 \). Suppose, to the contrary, \( \gamma(F_i) < 24/7 \).

An edge with one endvertex in \( X_i \) and the other one not in \( F_i \) is called an unused edge. The total number of unused edges will be noted by \( \Theta_i \). For convenience, we define \( a_i = |X_i| \) and \( b_i = |Y_i| \). Clearly, \( ||G[X_i]|| \geq a_i - 1 \) and \( \Theta_i \geq 0 \). Further,

\[
\Theta_i \geq (4a_i + t_3) - (2(t_1 + t_2) + 3t_3 + 2||G[X_i]|| + 4(a_i - \omega(X_i)))
\]

\[
= 4a_i - 2(t_1 + t_2 + t_3) - 2||G[X_i]|| - 4(a_i - \omega(X_i)).
\]
The above inequality comes from the following facts:

- The total degrees of vertices of $X_i$ is $4a_i + t_3$ by (F).
- The number of supporting edges with one endvertex in $X_i$ and the other one either an isolated vertex, a semi-isolated vertex, or an isolated edge in $G[V_3]$ is $2(t_1 + t_2)$.
- The number of supporting edges with one endvertex in $X_i$ and the other one at $P_3$ in $G[V_3]$ is $3t_3$.
- The total degree of $G[X_i]$ is at least $2a_i - 2$ since $G[X_i]$ is connected.
- If $v \in X_i$ is a descendant of a type II edge or $P_3$ in $G[V_3]$, we have $\omega(v) = 1/2$ or 0. If $\omega(v) = 1/2$, we have $d_H(v) \geq d_G(v) - 2 \geq 2$. If $\omega(v) = 0$, we only use the fact $d_H(v) \geq 0$. This is the reason of subtracting $4(a_i - \omega(X_i)) = 4(|X_i| - \omega(X_i))$.

And, by counting the degrees of vertices in $V_3$, the company vertices, and the edges incident to $X_i$, we have

$$\gamma(F_i) \geq \frac{3t_1 + 6t_2 + 9t_3 + 2\|G[X_i]\| + 2(t_1 + t_2) + 3t_3 + 2(t_2 + t_3) + \Theta_i}{t_1 + 2t_2 + 3t_3 + \omega(X_i) + (1/2)(t_2 + t_3)} \quad (1)$$

$$\gamma(F_i) = \frac{5t_1 + 10t_2 + 14t_3 + 2\|G[X_i]\| + \Theta_i}{t_1 + (5/2)t_2 + (7/2)t_3 + \omega(X_i)} \quad (2)$$

Since $G[X_i]$ is connected and there are $t_1 + t_2 + t_3$ primitive edges (some may be repeated), $a_i \leq t_1 + t_2 + t_3 + 1$ and $\|G[X_i]\| \geq a_i - 1$. The remainder of the proof will be divided into a few cases according to the value of $\omega(X_i)$.

**Case 0** Suppose that $\omega(X_i) \leq a_i - 1$.

Plugging $\omega(X_i) \leq a_i - 1$ in (2) and solving the following inequalities,

$$\frac{24}{7} \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2(a_i - 1)}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i - 1},$$

we obtain that $11t_1 + 10t_2 + 14t_3 < 10(a_i - 1)$ which contradicts that $t_1 + t_2 + t_3 \geq a_i - 1$.

**Case 1** Suppose that $\omega(X_i) = a_i - 1/2$.

Solving the inequalities

$$\frac{24}{7} \geq \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2(a_i - 1)}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i - 1/2},$$

we obtain that $11t_1 + 10t_2 + 14t_3 < 10a_i + 2$. Since $11t_1 + 10t_2 + 14t_3 \geq 10(t_1 + t_2 + t_3) \geq 10(a_i - 1)$, we have either $t_1 + t_2 + t_3 = a_i$ or $t_1 + t_2 + t_3 = a_i - 1$. If $t_1 + t_2 + t_3 = a_i - 1$, then,

$$2\|G[X_i]\| + \Theta_i \geq 2\|G[X_i]\| + (4a_i - 2(t_1 + t_2 + t_3) - 4(a_i - \omega(X_i))$$

$$- 2\|G[X_i]\|) = 2a_i.$$
By (2), we have
\[
\frac{24}{7} > \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2a_i}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i - 1/2}.
\]

Which implies that \(11t_1 + 10t_2 + 14t_3 < 10a_i - 12\), a contradiction. Thus, \(t_1 + t_2 + t_3 = a_i\).

If \(\|G[X_i]\| \geq a_i\), the same inequalities as above would be obtained, which leads, to a contradiction. Therefore, \(\|G[X_i]\| = a_i - 1\). So, \(G[X_i]\) is a tree. Let \(x\) and \(y\) be two distinct leaves of \(G[X_i]\). Since \(\omega(X_i) = a_i - 1/2\), only one vertex of \(X_i\) is a descendant. Without loss of generality, we assume that \(x\) is not a descendant obtained by splitting a vertex. Let \(xx^* \in E(G[X_i])\). Since \(\|G[X_i]\| = a_i - 1 = t_1 + t_2 + t_3 - 1\), we count \(xx^*\) at most two times as a primitive edge. Since \(d_G(x) \geq 4\), there is at least one unused edge incident with \(x\). In particular, we have \(\Theta_i \geq 1\). By (2), we have that
\[
\frac{24}{7} > \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2(a_i - 1) + 1}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i - 1/2},
\]

which implies that \(11t_1 + 10t_2 + 14t_3 < 10a_i - 5\), a contradiction to \(t_1 + t_2 + t_3 = a_i\).

**Case 2** Suppose that \(\omega(X_i) = a_i\).

Solving the inequalities
\[
\frac{24}{7} > \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2(a_i - 1)}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i},
\]

we obtain that \(11t_1 + 10t_2 + 14t_3 < 10a_i + 14\). Using the fact, \(t_1 + t_2 + t_3 \geq a_i - 1\), we have that \(t_1 + t_2 + t_3 = a_i - 1\), \(a_i\), or \(a_i + 1\). Further, if \(t_1 + t_2 + t_3 = a_i + 1\), we have \(t_3 = 0\) and \(t_1 \leq 3\), since \(11t_1 + 10t_2 + 14t_3 < 10a_i + 14\).

If \(t_1 + t_2 + t_3 = a_i - 1\), we have that
\[
2\|G[X_i]\| + \Theta_i \geq 2\|G[X_i]\| + (4a_i - 2(t_1 + t_2 + t_3) - 2\|G[X_i]\|) = 2a_i + 2.
\]

Plugging this in (2), we obtain that
\[
\frac{24}{7} > \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2a_i + 2}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i},
\]

which implies that \(11t_1 + 10t_2 + 14t_3 < 10a_i - 14\), a contradiction to that \(t_1 + t_2 + t_3 = a_i - 1\). If \(t_1 + t_2 + t_3 = a_i\), we have that
\[
2\|G[X_i]\| + \Theta_i \geq 2\|G[X_i]\| + (4a_i - 2(t_1 + t_2 + t_3) - 2\|G[X_i]\|) = 2a_i.
\]
Plugging this in (2), we obtain that

\[
\frac{24}{7} \geq \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2a_i}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i},
\]

which implies that \(11t_1 + 10t_2 + 14t_3 < 10a_i\), a contradiction to \(t_1 + t_2 + t_3 = a_i\). Thus, \(t_1 + t_2 + t_3 = a_i + 1\).

Recall, in this case, we have that \(t_3 = 0\) and \(t_1 \leq 3\). If \(2||G[X_i]|| + \Theta_i \geq 2a_i - 1\), 2 gives us that

\[
\frac{24}{7} > \gamma(F_i) \geq \frac{5t_1 + 10t_2 + 14t_3 + 2a_i - 1}{t_1 + (5/2)t_2 + (7/2)t_3 + a_i},
\]

which implies that \(11t_1 + 10t_2 + 14t_3 < 10a_i + 7\), a contradiction to \(t_1 + t_2 + t_3 = a_i + 1\). Hence, \(||G[X_i]|| = a_i - 1\) and \(\Theta_i = 0\). So, \(G[X_i]\) is a tree. Since \(t_1 + t_2 + t_3 = a_i + 1\) and \(||G[X_i]|| = a_i - 1\), when we count the edges of \(G[S]\) as primitive edges, we have the following three cases.

- No edges have been counted more than once as primitive edges.
- No edges have been counted more than once as primitive edges except two, with each one counted at most twice.
- No edges have been counted more than once as primitive edges except two, with each one counted at most three times.

If \(a_i \geq 3\), let \(u\) and \(x\) be two leaves of \(G[X_i]\) with edges \(uv, xy \in E(G[X_i])\). Note that \(v\) and \(y\) may be the same vertex. Thus, either \(xy\) or \(uv\), are counted no more than twice as primitive edges. Thus, there is an unused edge incident with \(u\) or there is an unused edge incident with \(x\), since \(d_G(u) \geq 4\) and \(d_G(x) \geq 4\). This implies that \(\Theta_i \geq 1\), a contradiction to that \(\Theta = 0\). Then, \(a_i = 2\). Let \(S_i = \{x, y\}\). Since \(\Theta_i = 0\), \(xy\) must be counted three times as primitive edges and \(d_G(x) = d_G(y) = 4\). Since \(t_3 = 0\), \(F_i\) is one of the forbidden graphs in 3.7, a contradiction.

**REFERENCES**


