SATURATION NUMBERS FOR TREES

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Abstract

For a fixed graph $H$, a graph $G$ is $H$-saturated if there is no copy of $H$ in $G$, but for any edge $e \notin G$, there is a copy of $H$ in $G + e$. The collection of $H$-saturated graphs of order $n$ is denoted by $\text{SAT}(n, H)$, and the saturation number, $\text{sat}(n, H)$, is the minimum number of edges in a graph in $\text{SAT}(n, H)$. The saturation numbers $\text{sat}(n, T)$ for some families of trees will be determined precisely. Some classes of trees for which $\text{sat}(n, T) < n$ will be identified, and trees $T$ in which graphs in $\text{SAT}(n, T)$ are forests will be presented. Also, families of trees in which $\text{sat}(n, T) \geq n$ will also be presented. The maximum and minimum values of $\text{sat}(n, T)$ for the class of all trees will be given. Some properties of $\text{sat}(n, T)$ and $\text{SAT}(n, T)$ for trees will be discussed.
1 INTRODUCTION

Only finite graphs without loops or multiple edges will be considered. Notation will be standard, and generally follow the notation of Chartrand and Lesniak [CL05]. For a graph $G$ we use $G$ to represent the vertex set $V(G)$ and the edge set $E(G)$ when it is clear from the context.

For a fixed graph $H$, a graph $G$ is $H$-saturated if there is no copy of $H$ in $G$, but for any edge $e \not\in G$, there is a copy of $H$ in $G + e$. The collection of $H$-saturated graphs of order $n$ is denoted by $\text{SAT}(n, H)$, and the saturation number, denoted $\text{sat}(n, H)$, is the minimum number of edges in a graph in $\text{SAT}(n, H)$. The maximum number of edges in a graph in $\text{SAT}(n, H)$ is the well known Turán extremal number (see [Tur41]), and is usually denoted by $\text{ex}(n, H)$. The graphs in $\text{SAT}(n, H)$ with a minimum number of edges will be denoted by $\text{SAT}(n, H)$, and those with a maximum number of edges will be denoted by $\overline{\text{SAT}}(n, H)$. Thus, all graphs in $\text{SAT}(n, H)$ have $\text{sat}(n, H)$ edges and graphs in $\overline{\text{SAT}}(n, H)$ have $\text{ex}(n, H)$ edges.

The notion of the saturation number of a graph was introduced by Erdos, Hajnal, and Moon in [EHM64] in which the authors proved $\text{sat}(n, K_t) = \binom{t-2}{2} + (n-t+2)(t-2)$ and $\overline{\text{SAT}}(n, K_t) = \{K_{t-2} + K_{n-t+2}\}$. Since then $\text{sat}(n, G)$ and $\overline{\text{SAT}}(n, G)$ have been investigated for a range of graphs $G$. Some examples include cycles, bipartite graphs, matchings, friendship graphs, and books. The exact value of $\text{sat}(n, G)$ and a complete characterization of $\overline{\text{SAT}}(n, G)$ are known for very few graphs $G$. For a summary of known results see [FFS08]. Generalizations to hypergraphs also exist. See [Pik04].

The emphasis of this paper will be on exploring $\text{sat}(n, T)$ when the graph $T$ is a tree. For special trees, specifically paths and stars, the saturation numbers are already known. The known results will be discussed in Section 2. The saturation numbers $\text{sat}(n, T)$ for some families of trees such as when $T$ is a broom, double broom, or caterpillars will be determined precisely. The class of trees for which $\text{sat}(n, T) < n$ will be explored, and large classes of trees will be shown to have this property. Such trees are said to have “small” saturation numbers, and these are trees $T$ in which graphs in $\text{SAT}(n, T)$ are forests. Also, families of trees in which $\text{sat}(n, T) \geq n$ will also be studied. The maximum and minimum values of $\text{sat}(n, T)$ for the class of all trees will be determined. Some properties of $\text{sat}(n, T)$ and $\overline{\text{SAT}}(n, T)$ for trees will be discussed, and a comparison of the properties of $\text{ex}(n, T)$ and $\text{sat}(n, T)$ will be made.
2 KNOWN RESULTS

In [KT86] Kászonyi and Tuza proved several general results on saturated graphs including the best known general upper bound. The results particularly relevant here are those concerning trees, specifically stars and paths which are summarized below. The star on $k$ vertices will be denoted $S_k = K_{1,k-1}$ and $P_k$ denotes the path on $k$ vertices.

**Theorem 1.** (a) $\text{sat}(n, S_{k+1}) = \begin{cases} \binom{k}{2} + \binom{n-k}{2} & \text{if } k + 1 \leq n \leq k + \frac{k}{2} \\ \left\lceil \frac{k-1}{2} n - \frac{k^2}{8} \right\rceil & \text{if } k + \frac{k}{2} \leq n \end{cases}$

(b) Let $a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if } k = 2t \\ 4 \cdot 2^{t-1} - 2 & \text{if } k = 2t + 1 \end{cases}$

If $n \geq a_k$ and $k \geq 6$, then $\text{sat}(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$.

(c) For any tree $T$ on $k$ vertices such that $T \neq S_k$, $\text{sat}(n, T) < \text{sat}(n, S_k)$

Note that $\text{sat}(n, P_i)$ and $\text{SAT}(n, P_i)$ for $i \leq 5$ are also established in [KT86].

**Theorem 2.** $\text{SAT}(n, S_k) = \begin{cases} K_{k-1} \cup K_{n-k+1} & \text{if } k \leq n \leq \frac{3k-3}{2} \\ G' \cup K_p & \text{if } \frac{3k-3}{2} \leq n \end{cases}$

where $p = \lfloor k/2 \rfloor$ and $G'$ is a $(k-1)$-regular graph on $n-p$ vertices. Note that in the case when $n \geq \frac{3k-3}{2}$, an edge is added if $k-1$ and $n-p$ are both odd.

The set $\text{SAT}(n, P_k)$ is more complicated and may contain many nonisomorphic graphs. But, all minimally $P_k$-saturated trees have a common structure which will be useful in later proofs. Thus, these “almost binary” trees, as Kászonyi and Tuza call them, are emphasized.

[Figure 1: $T_6$ and $T_7$]

Define $T_k$ to be the “almost binary” tree with $\lfloor k/2 \rfloor$ levels such that all vertices except end vertices have degree 3. Note that the top level will have one or two vertices depending on whether $k$ is even or odd. (See Fig. 2).
Not only is $T_k$ a $P_k$-saturated tree but the addition of any number of pendant vertices to those vertices already adjacent to vertices of degree 1 does not change this property. In the theorem below, observe that $a_k = |V(T_k)|$.

**Theorem 3.** Let $P_k$ be a path on $k \geq 3$ vertices and let $T_k$ be the tree defined above. Let $a_k = \begin{cases} 3 \cdot 2^{m-1} - 2 & \text{if } k = 2m \\ 4 \cdot 2^{m-1} - 2 & \text{if } k = 2m + 1 \end{cases}$ Then, for $n \geq a_k$, $SAT(n, P_k)$ consists of a forest with $\lfloor n/a_k \rfloor$ components. Furthermore, if $T$ is a $P_k$-saturated tree, then $T_k \subseteq T$.

### 3 MINIMUM SATURATION NUMBERS FOR TREES

Recall that Kaszonyi and Tuza showed that, of all trees on $k$ vertices, the one with largest saturation number is the star and it is unique. Curiously, we will show that the unique tree on $k$ vertices with smallest saturation number is almost the same graph, a star with a single subdivided edge.

For $k \geq 4$ let $T^*_k$ be the tree on $k$ vertices obtained by subdividing one edge of a star on $k-1$ vertices. Thus, $T^*_k$ has a vertex of degree $k-2$ and a vertex of degree 2 with the remaining vertices of degree 1 (See Figure 3.) We will show that graphs in $SAT(n, T^*_k)$ look like a forest $\lfloor n/k \rfloor$ stars with possibly an additional $K_2$.

![Figure 2: $T^*_6$](image)

Specifically, let $F$ be the forest on $n$ vertices equal to

1. $(n/k)S_k$ if $n \equiv 0 \mod k$,
2. $((n-k-1)/k)S_k \cup S_{k+1}$ if $n \equiv 1 \mod k$,
3. $((n-k-p)/k)S_k \cup K_2 \cup S_{k+p-2}$ if $n \equiv p \mod k$ for $p \geq 2$.

Thus, $F$ has $\lfloor (n+k-2)/k \rfloor$ tree components and $n - \lfloor (n+k-2)/k \rfloor$ edges. It is obvious that $F$ does not contain $T^*_k$ as a subgraph, and it is also clear that the addition of any edge to $F$ will produce a copy of $T^*_k$. Hence, $F \in SAT(n, T^*_k)$. First we will show that $F \in SAT(n, T^*_k)$ and $\text{sat}(n, T) > |E(F)|$ for all $k$-vertex trees $T \neq T^*_k$. 

4
Lemma 1. Let $T$ be a tree of order $k$ and let $T'$ be a tree of order $k$ in $\text{SAT}(n, T)$. Then, $k \geq 4$, $T' = S_k$ and $T = T^*_k$.

Proof. Small values of $k$ can be easily checked. For example for $k = 5$, there exist only two trees that are not stars and neither is $T$-saturated for any tree on 5 vertices.

Thus, we assume $k \geq 6$ and that $T'$ is not a star, $S_k$. Thus, $T'$ contains a path with at least 4 vertices. Also, observe that since $T'$ is $T$-saturated and both have the same order, for every edge $e \notin T'$ there exists an edge $e' \in T'$ such that $T' + e - e' = T$. Select a longest path $P$ in $T'$, say $P = (x_1, x_2, \ldots, x_{q-1}, x_q)$ with $q \geq 4$.

Case 1: $\deg(x_2) \geq 3$ or $\deg(x_{q-1}) \geq 3$

Without loss of generality, assume $\deg(x_2) \geq 3$. Let $y$ be another end vertex adjacent to $x_2$. Let $e = x_1y$. Then without loss of generality, $e' = yx_2$ and we conclude that $T = T' + e - e'$ and, in particular, has precisely one fewer vertex of degree 1 than $T'$. Thus every pair of vertices in $T'$ of degree 2 or more are adjacent. But this forces $T'$ to be a symmetric double star. (That is, $T'$ consists of two stars of the same order joined by an edge between their centers.) But now, if let $e$ be an edge between end vertices on different stars, there is no edge $e'$ whose deletion will produce a tree isomorphic to the one obtained when $e$ is between end vertices on the same star. So $T'$ is not $T$-saturated for any tree, a contradiction.

Case 2: $\deg(x_2) = \deg(x_{q-1}) = 2$

Let $e = x_1x_3$. Then, in order to avoid a copy of $T$ in $T'$, $e' = x_1x_2$. So, $T = T' + e - e'$ and therefore $T$ must have exactly one more vertex of degree 1 than $T'$. But $T$ has three more vertices of degree 1 than $T' + \{x_2, x_q\}$ and there is no edge whose deletion will produce three additional vertices of degree 1, a contradiction.

So, $T'$ cannot contain a path of four or more vertices and is therefore a star. Finally, $T^*_k$ is the only tree $T$ for which $T'$ is $T$-saturated. 

Theorem 4. For any tree $T_k$ of order $k \geq 5$ and any $n \geq k + 2$,

$$\text{sat}(n, T_k) \geq n - \lfloor (n + k - 2)/k \rfloor.$$ 

Moreover, $T^*_k$ is the only tree attaining this minimum for all $n$.

Proof. Let $G \in \text{SAT}(n, T_k)$ for a fixed tree $T_k$ of order $k \geq 5$. Observe that any component of $G$
of order less than \( k \) must be complete and any pair of components must contain at least \( k \) vertices. Since \( k \geq 5 \), this implies \( G \) can have at most one component of the form \( K_i \) for \( i \in \{1, 2\} \).

Thus, if \( T_k \neq T^*_k \) then Lemma 1 implies that any tree components of \( G \) have order at least \( k + 1 \) with the possible exception of a single component of order 2 or less. Thus \( sat(n, T_k) = |E(G)| \geq n - \lfloor \frac{n-1}{k+1} \rfloor - 1 \geq n - \lfloor \frac{n+k-2}{k} \rfloor \). Furthermore, the previous inequality is strict for \( n \geq k^2 + k + 2 \).

Assume \( T_k = T^*_k \). If \( |E(G)| < n - \lfloor (n + k - 2)/k \rfloor \), then \( G \) has more than \( \lfloor (n + k - 2)/k \rfloor \) components. Thus, at least two of them have order strictly less than \( k \). So they are both complete and together contain at least \( k \) vertices. Hence, we could replace these two components with a star on the same number of vertices to create a new graph \( G' \) that is \( T^*_k \)-saturated but with fewer edges, contradicting the assumption \( G \in \text{SAT}(n, T^*_k) \). So, \( sat(n, T^*_k) = n - \lfloor (n + k - 2)/k \rfloor \).

The following corollary follows immediately from the preceding proof.

**Corollary 1.** Every graph \( G \in \text{SAT}(n, T^*_k) \) is a forest of \( \lfloor (n + k - 2)/k \rfloor \) stars. If \( n - k\lfloor n/k \rfloor \geq 2 \), then one of the stars is \( K_2 \).

**QUESTION TO ME:** Does there exist any tree with saturation number between the star and the evenly divided double broom? That is: Can there exist \( T \) where

\[
sat(n, S_k) = (k - 2)n/2 - C > sat(n, T) > (k - 2)n/4 - C' = sat(n, B_{k/2,k/2})?
\]

### 4 SUBTREE PROPERTIES

There are no general monotone properties for subtrees of trees relative to the function \( sat(n, T) \). The following theorems verify this in a very strong way.

We introduce some useful notation. Let \( G \) be a nonregular graph. Define \( \delta_2(G) \) to be the second smallest possible degree of a vertex in the graph \( G \). That is, \( G \) contains a vertex \( x \) such that \( \deg(x) = \delta_2(G) > \delta(G) \) but there does not exist a vertex \( z \) such that \( \delta(G) < \deg(z) < \delta_2(G) \).

**Theorem 5.** If \( T \) is a tree of order at least 5 such that \( \delta_2(T) = d \), then \( sat(n, T) \geq \frac{d-1}{2}n - \frac{d^2}{8} \).

Proof: Let \( G \in \text{SAT}(n, T) \). Then any two vertices of degree \( d-2 \) or less must be adjacent. Let \( s \) be the number of vertices of degree \( d-2 \) or less. Then \( |E(G)| \geq \binom{s}{2} + (n-s)(d-1)/2 \geq \frac{d-1}{2}n - \frac{d^2}{8} \).

\[ \square \]
**Corollary 2.** Every tree $T$ is the subtree of a tree $T'$ such that $\text{sat}(n, T') \geq Cn$ for any constant $C$ and $n$ sufficiently large.

Proof: Let $d = 2C + 2$. Construct a tree $T'$ such that $\delta_2(T') \geq d$ by adding pendant vertices to those vertices of $T$ with degree 2 or more. \hfill \Box

On the other hand we will now show that every tree $T$ is the subtree of a tree $T'$ such that $\text{sat}(n, T') < n$. We first need to prove a structural lemma. To do this we need to introduce some specialized definitions.

Let $T_{m,d}$ be an almost $d$-ary tree with $\lceil \frac{m}{2} \rceil$ levels (See figure 4.) where every vertex has degree $d + 1$ except those at the lowest level (which have degree 1). Furthermore, the top level will have one vertex if $m$ is even and two adjacent vertices if $m$ is odd. So $T_{m,d}$ has $1 + (d + 1) \frac{d \frac{m-2}{2} - 1}{d - 1}$ vertices if $m$ is even and $2 \frac{d \frac{m-1}{2} - 1}{d - 1}$ if $m$ is odd. Note that this definition is similar to the almost binary trees defined in the Kaszonyi and Tuza paper.

![Image](image_url)

**Figure 3:** $T_{6,4}$ and $T_{7,4}$

We will label the levels of this tree from the bottom up. The vertices on the top level, we will label $r$ for root if $m$ is even and $r_1$ and $r_2$ if $m$ is odd. Finally, we will refer to the $d + 1$ subtrees below $r$ (or the $d$ subtrees below $r_i$) to mean the $d + 1$ trees that would result from the deletion of the edges incident to $r$ (or the $d$ trees resulting from the deletion of edges incident to $r_i$). Note that each of these is a true $d$-ary tree.

**Lemma 2.** Given any edge $e \not\in E(T_{m,d})$, the almost $d$-ary tree defined above, there exists a path in $T_{m,d} + e$ on

(a) $\frac{m}{2} + 1$ vertices beginning at $r$ and using vertices from at most two of the subtrees under $r$ for $m$ even

or

(b) $\frac{m-1}{2} + 2$ vertices ending at $r_2$ and using vertices from at most one of the $d$ subtrees under $r_2$.\hfill 7
Proof. Let $T_{m,d} = T$ be the almost $d$-ary tree defined above. Assume $m$ is even. Let $e = yz$ be an edge not in $T$ and assume the level of $y$ is less than or equal to that of $z$.

**Case 1:** $z$ lies on the unique $ry$ path ($r = z$ is allowed)

We construct the path on $m/2 + 1$ vertices as follows. Starting at $r$, take the unique path in $T$ down to $z$, take edge $e = zy$, take the unique path from $y$ up to $z_c$, a child of $z$, and from $z_c$ take a path down to any end vertex. Recall that $z_c$ will have $d - 1$ children other than $y$ from which to choose. Since this path includes a path from $r$ down to an end vertex (through $z$ and $z_c$) and at least one additional vertex, namely $y$, it must contain at least $m/2 + 1$ vertices. Also, observe that this path uses at most one of the subtrees under $r$, namely the one containing the unique $ry$ path.

**Case 2:** $z$ does not lie on the unique $ry$ path

Construct the desired path as follows. Starting at $r$, take the unique path down to $y$, take edge $e = yz$, take any path from $z$ down to an end vertex. Since $z$ is on a level at least as high as $y$, the path contains at least two vertices from the same level and therefore at least $m/2 + 1$ vertices. Also, observe that it uses at most two subtrees under $r$, namely the one containing $y$ and the one containing $z$ (which may in fact be the same).

Assume $m$ is odd. Let $e = yz$ be an edge not in $T$.

**Case 1:** the edge $e$ lies entirely in the subtree rooted by $r_1$ or the subtree rooted by $r_2$

Without loss of generality, assume $e$ lies entirely in the tree rooted by $r_1$. Then applying the method when $m$ is even, we know there exists a path on at least $(m - 1)/2 + 1$ vertices starting at $r_1$ and completely contained in this subtree. Add edge $r_1r_2$ and the desired path is obtained using no subtree under $r_2$.

**Case 2:** the edge $e$ contains one vertex from the tree rooted at $r_1$ and one from the tree rooted at $r_2$

Without loss of generality, assume $y$ in the subtree rooted at $r_1$, $z$ is in the subtree rooted at $r_2$, and that the level of $y$ is no more than that of $z$. (Note that it is possible that $z = r_2$.) Then construct the desired path by starting at $r_2$, going down to $z$, taking edge $zy$, take the path from $y$ up to $r_1$, and finally take a path from $r_1$ down to any end vertex (that doesn't require using vertex $y$.) Observe that this contains a path from $r_2$ to an end vertex under $r_1$ plus at least one additional vertex, namely $y$. Thus it must contain at least $(m - 1)/2 + 2$ vertices and it uses vertices from at most one subtree under $r_2$, namely the one containing $z$. \qed
Theorem 6. Let $T$ be a tree with maximum degree $d \geq 3$, diameter $m \geq 4$, and such that there exists a diameter path $P$ in $T$ such that the first $\lceil m/2 \rceil$ vertices on this path have degree 2 or less in $T$. Then $T_{m,d}$ is $T$-saturated and $\text{sat}(n, T) < n$ for $n \geq |E(T_{m,d})|$.

Proof: Since $T_{m,d}$ has no path on $m$ vertices, it is $T$-free. Now consider $T_{m,d} + e$ for some new edge $e$. From the previous lemma, we know there exists a path, $Q$ on at least $\lceil m/2 \rceil + 1$ vertices that ends in a vertex in the top level of $T_{m,d}$ (either $r$ or $r_2$ depending on the parity of $m$). Additionally, this top vertex has at least $d - 1$ subtrees under it all of which are disjoint from $Q$. Thus, $T_{m,d}$ must contain a copy of $T$. The upper bound for the saturation number of $T$ follows immediately from the observation that, for $n$ sufficiently large, there exists a $T$-saturated forest. □

Corollary 3. Every tree $T$ is the subtree of a tree $T'$ such that $\text{sat}(n, T') < n$.

Proof: Given tree $T$ with diameter $p$, construct $T'$ by adding to $T$ a path on $p + 1$ vertices and apply the previous theorem with diameter $m = 2p$. □

It should also be observed that the proof of lemma 2 implies that, for $m$ odd and any new edge $e$, one can find a path on $m$ vertices in $T_{m,d}$ such that the middle vertex (vertex $\lceil m/2 \rceil$ on the path) is in one of the top two levels. Thus, the theorem above can be extended to include trees for which the degree of the middle vertex on a diameter path is greater than 2 provided the longest path starting at this middle vertex away from the diameter path contains at most $(m - 3)/2$ vertices.

On the other hand, for $m$ even, by considering an edge $e$ from the top level to the bottom level, we see that we cannot avoid forcing a vertex of degree two close to the middle ($\lfloor m/2 \rfloor$). Furthermore, by adding the edge from level $r$ to an end vertex directly under it, we see we cannot avoid a vertex of degree 2 in vertex $r$ of the longest path.

5 Some Results Concerning Specific Trees

The following technical lemma simplifies the proof of the saturation number in many cases.

Lemma 3. Assume $H$ is a graph of order at least 5 and $T$ is an $H$-saturated tree such that

(a) for every $H$-saturated tree $T'$, then $|V(T')| \geq |V(T)|$,
(b) for every \( k, 1 \leq k \leq |V(T)| - 1 \), there exists an \( H \)-saturated tree \( T_k \) of order \( |V(T)| + k \), and

(c) the union of any pair of trees in the set \( S = \{ T, T_1, T_2, \ldots, T_{|V(T)|-1} \} \) is \( H \)-saturated, then for \( n \geq |V(T)| \),

(1) there exists a graph \( G \in \text{SAT}(n, H) \) such that \( G \) is a forest

and

(2) \( n - \left\lceil \frac{n-1}{|V(T)|} \right\rceil - 1 \leq \text{sat}(n, H) \leq n - \left\lfloor \frac{n}{|V(T)|} \right\rfloor \)

**Proof.** (Part 1) For any \( n \geq |V(T)| \) there exists an \( H \)-saturated forest consisting of \((\left\lceil \frac{n}{|V(T)|} \right\rceil - 1)T \cup T_p \) where \( n \equiv p \mod |V(T)| \). Call this graph \( G' \). Let \( G \in \text{SAT}(n, H) \) with the minimum number of components that are not trees. Let \( A \) be the set of vertices of \( G \) in components that are not trees. Then the graph \( G - A = F \) is a (nonempty) forest. Either \( F = K_1 \) or \( F = K_2 \) or \( F \) contains a tree on at least \( |V(T)| \) vertices and thus we can assume all such “large” trees are elements from \( S \). If \( F = K_1 \) or \( F = K_2 \), then \( |E(G)| \geq n - 1 \geq |E(G')| \). If \( F \) contains a large tree, then the vertices of \( A \) along with vertices of the large tree can be replaced entirely with elements from \( S \), forming an element of \( \text{SAT}(n, H) \) with fewer nontree components.

(Part 2) The upper bound is obtained from graph \( G' \) described earlier. The lower bound results from the observation that a minimally \( H \)-saturated forest might have \( K_1 \) or \( K_2 \) as a component. ❑

**QUESTION TO ME:** If \( G \) and \( G' \) are both \( H \)-saturated, is \( G \cup G' \) \( H \)-saturated?

**BROOMS**

First we will consider brooms, denoted \( B_{r,k} \), where \( r \) corresponds to the number of vertices on the handle and \( k \) denotes the number of bristles. (See figure 5.) The vertex of degree \( k + 1 \) will be referred to as the center of the broom. Obviously, \( B_{r,k} \) contains \( r + k \) vertices. Observe that one of the interesting properties of the collection of all brooms is that it contains all of the trees for which the saturation number is, thus far, known exactly: the star \( B_{1,k} \), the path \( B_{r,1} \), and the star with one subdivided edge \( B_{3,k} \). In the theorems below, we will find the saturation number for some specific brooms.

**Theorem 7.** For \( k \geq 2 \) and \( n \geq 2k + 5 \), \( \text{sat}(n, B_{4,k}) = n - \left( \left\lfloor \frac{n-2}{2k+3} \right\rfloor + 1 \right) \).
Proof. Define the double star, $S_{a,b}$, to be the graph obtained by adding the edge between the centers of the stars, $S_a$ and $S_b$. See figure 5. Note $S_{k+2,k+1}$ is $B_{4,k}$-saturated. In addition, $S_{a,b}$ is $B_{4,k}$-saturated for any $a \geq k+2$, $b \geq k+1$.

Figure 4: $B_{5,3}$

Figure 5: $S_{6,4}$

In order to apply lemma 3, we need to show that every $B_{4,k}$-saturated tree has at least $|V(S_{k+2,k+1})| = 2k+3$ vertices. Let $T$ be any $B_{4,k}$-saturated tree. By adding an edge between two vertices of degree 1 in $T$, we conclude $T$ must have at least one vertex of degree at least $k+1$.

Assume $T$ has precisely one vertex of degree at least $k+1$, say $x$. Then $x$ cannot be adjacent to two vertices of degree 1 since the edge between them would not produce a $B_{4,k}$. Let $y$ be a neighbor of $x$ of degree at least 2. Then, the length of the longest path in $T$ starting at $x$ and using edge $xy$ is exactly 2. Furthermore, adding the edge between $x$ and the end vertex of such a path implies that $\deg(y) \geq 3$. Thus, if $T$ has precisely one vertex of high degree, $|V(T)| \geq 3k+2 \geq 2k+3$.

Assume $T$ has at least two vertices of degree $k+1$ or more. Then, to avoid a $B_{4,k}$, these vertices must be adjacent. Thus, there are precisely two vertices of high degree. Finally, by adding the edge between a neighbor of one vertex of high degree to a neighbor on the other (both of which must have degree 1), we conclude at least one of the high degree vertices must have degree at least $k+2$. So, every nontrivial tree component $T$ must have at least $2k+3$ vertices.

Finally, $S_{k+2,k+1} \cup K_2$ is $B_{4,k}$-saturated. But, by the argument above, there does not exist any tree $T$ such that $T \cup K_1$ is $B_{4,k}$-saturated. Thus, for $n \geq 2k+5$, the graph consisting of a disjoint union of one $K_2$ and $\lfloor \frac{n-2}{2k+3} \rfloor$ double stars each of which has $S_{k+2,k+1}$ as a subgraph is a minimally $B_{4,k}$-saturated graph and $\text{sat}(n, B_{4,k}) \leq n - \left(\lfloor \frac{n-2}{2k+3} \rfloor + 1\right)$.

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Theorem 8. For $k \geq 2$ and $n \geq 2k + 6$, sat$(n, B_{5,k}) = n - \left\lfloor \frac{n}{2k+6} \right\rfloor$

Proof. Recall that $T_6$ is the almost binary tree which is a subtree of any $P_6$-saturated tree. (See Figure 2.) Let $u$ and $v$ be any two of the three vertices of $T_6$ in the middle level (those adjacent to vertices of degree 1.) Define $T_6^k$ to be the tree constructed from $T_6$ by adding $k - 2$ pendant vertices to each of $u$ and $v$. (See figure 5.) Note $T_6^k$ has two vertices of degree $k+1$ and is still $P_6$-saturated.

It is easy to check that for every new edge $e$, the graph $T_6^k + e$ contains a $P_6 = v_1v_2\cdots v_6$ such that $\deg_{T_6^k}(v_2) = k + 1$ and so that $|N(v_2) \cap V(P_6)| = 2$. So $T_6^k$ and $T_6^k \cup T_6^k$ are $B_{5,k}$-saturated. Furthermore, any number of additional pendant vertices can be added to any of the vertices in the middle level of $T_6^k$ and it will still be $B_{5,k}$-saturated.

Next we will show that if $T$ is a $B_{5,k}$-saturated tree, then $|V(T)| \geq 2k + 6$. By adding the edge between any two vertices of degree 1, we see $T$ must contain a vertex of degree at least $k + 1$.

![Figure 6: $T_6^4$](image)

Assume $T$ has precisely one vertex of degree $k + 1$ or more, say $x$. At most one of its neighbors can have degree 1. Thus, let $y$ be a neighbor of $x$ of degree at least 2. If the longest path starting at $x$ and proceeding through $y$ is of length 2, then adding the edge from $x$ to the end vertex of this path cannot produce a $B_{5,k}$. Thus, the longest path in $T$ starting at $x$ and using edge $xy$ has exactly 4 vertices. Furthermore, adding the edge between $x$ and the end vertex of such a path implies that the component of $T - x$ containing $y$ has at least 4 vertices. Thus, if $T$ has precisely one vertex of high degree, $|V(T)| \geq 4k + 2 \geq 2k + 6$.

Next assume $T$ has at least two vertices of degree $k + 1$ or more. Find two high degree vertices farthest apart, say $x_1$ and $x_2$. Then $d(x_1, x_2) \leq 2$. If $x_1x_2 \in E(T)$, then neither can be adjacent to two vertices of degree 1. Let $y$ be a neighbor of $x_1$ of degree at least 2 ($y \neq x_2$.) Adding the edge between $x$ and an end vertex of a longest path through $y$ implies $\deg(y) \geq 3$. Applying symmetry, we find $|V(T)| \geq 2k + 2 + 4(k - 1) \geq 2k + 6$. If $d(x_1, x_2) = 2$, then label as $y$ the vertex on the
path between them. Now all neighbors of $x_i$ other than $y$ must have degree 1. The copy of $B_{5,k}$ in $T + \{x_1x_2\}$ requires $T$ to have a path on three vertices starting at $y$ and disjoint from $x_1$ and $x_2$. Call it $yy_1y_2$. Now $\deg(y_2) = 1$. Adding the edge $yy_2$ forces $\deg(y_1) \geq 3$. Thus, $|V(T)| \geq 2k + 6$.

Thus, we have shown that all $B_{5,k}$-saturated trees have at least $2k + 6$ vertices.

Finally, observe that for any $B_{5,k}$-saturated tree $T$, the graph $T \cup K_i$ is not $B_{5,k}$-saturated for $i = 1, 2$ since adding an edge between $K_i$ and vertex of degree $k + 1$ in $T$ cannot produce a copy of $B_{5,k}$. Thus, by applying Lemma 3 we know that $\text{SAT}(n, B_{5,k})$ contains a forest each component of which has $T_k^6$ as a subgraph and $\text{sat}(n, B_{5,k}) = n - \lfloor \frac{n}{2k+6} \rfloor$.

REMINDER QUESTION TO ME: What exactly can we say about $\text{SAT}(n, B_{5,k})$ and $\text{SAT}(n, B_{4,k})$?

Theorem 9. For $n \geq a_{r+1}$ and $r \geq 4$, then $\text{sat}(n, B_{r,2}) = \text{sat}(n, P_{r+1})$ where

$$a_k = \begin{cases} 
3 \cdot 2^{m-1} - 2 & \text{if } k = 2m \\
4 \cdot 2^{m-1} - 2 & \text{if } k = 2m + 1 
\end{cases}$$

Proof. Recall that the structure of minimally $P_k$-saturated graphs is known (see [KT86] or Theorem 3 in the introduction). In particular they consist of forests such that all trees contain a common minimal subtree on $a_k$ vertices called “almost” binary trees and labeled $T_k$. It is easy to verify that these graphs are also $B_{k-1,2}$-saturated. Thus, we need to show that every $B_{r,2}$-saturated tree $T$ contains at least $a_{r+1}$ vertices. We will do this by showing that every $B_{r,2}$-saturated tree must be $P_{r+1}$-saturated.

Assume there exists a $B(r,2)$-saturated graph, $T$, that contains a path on $r + 1$ vertices. Let $P = x_1, x_2, \ldots, x_s$ be a longest path in $T$. So $s \geq r + 1$. Since $P$ is not itself $B(r,2)$-saturated, $P$ must contain a vertex of degree at least 3. Let $x_{i_0}$ be the first vertex on $P$ of degree 3 or more. Then $3 \leq i_0 \leq r - 1$.

Let $e = x_1x_{i_0}$ and let $c$ be the vertex of $T$ that is the center of the copy of $B_{r,2}$ in $T + e$. (That is, $\deg_{B_{r,2}}(c) = 3$.) We know $c \neq x_i$ for $1 \leq i \leq i_0 - 1$ because all these vertices have degree 2 in $T + e$. But for every other choice of $c$, the edge $e$ must appear in a path on $r$ vertices starting at $c$ which immediately implies $B_{r,2} \subset T$, a contradiction. So the longest path in any $B_{r,2}$-saturated tree $T$, is at most $r$. Thus, $T$ is $P_{r+1}$ saturated and therefore contains at least $a_{r+1}$ vertices.

QUESTION: What to say about the set of minimally $B_{r,2}$-saturated graphs.
DOUBLE STARS

Let $S_{t,r}$ be the graph on $t + r$ vertices constructed by adding the edge between the centers of a star on $t$ vertices and a star on $r$ vertices. (See figure 5.)

**Theorem 10.** For $n \geq 6$, $sat(n, S_{3,3}) = \begin{cases} n & n \equiv 0 \mod 3 \\ n + 1 & otherwise \end{cases}$.

**Proof.** If $n \equiv 0 \mod 3$, then the graph consisting of $n/3$ disjoint triangles provides an upper bound for $sat(n, S_{3,3})$. If $n \equiv 1 \mod 3$, the graph consisting of the union of $(n - 7)/3$ disjoint triangles and one component on 7 vertices consisting a two triangles connected by a single path of length 2 provides an upper bound for $sat(n, S_{3,3})$. The example in the case that $n \equiv 2 \mod 3$ consists of the union of $(n - 8)/3$ disjoint triangles and one component on 8 vertices consisting a 4-cycle and a 5-cycle which share a single vertex.

Let $G$ be a minimally $S_{3,3}$-saturated graph. We want to show that $|E(G)| \geq n$ or $n + 1$ accordingly. First, observe that $G$ cannot have two nonadjacent vertices of degree 1 since adding the edge between them cannot produce a copy of $S_{3,3}$. Also, if a component of $G$ is a cycle, it must be a 3-cycle since larger cycles are not $S_{3,3}$-saturated. Thus, if $\delta(G) \geq 2$ the result follows.

Assume $G$ contains a vertex $x$ such that $\deg(x) \leq 1$.

**case 1:** $x \in V(K_1)$ or $x \in V(K_2)$

Let $C$ and $C'$ be distinct components of $G$ such that $x \in C$. Note, it is enough to show that the average degree of vertices in $G$ is strictly greater than 2. By adding edges from $x$ to vertices in $C'$, it follows that $\delta(C') \geq 2$, and $C'$ must have at least two vertices of degree 3 or more that are adjacent and that share a common neighbor. If $G$ has three or more components, the result follows. If $G = C \cup C'$ the average degree of $G$ will be above 3 unless $C'$ has exactly two vertices of degree 3 and all others have degree 2. This forces $C'$ to be a cycle with a single chord or two cycles connected by a path. In neither case is the graph $S_{3,3}$-saturated.

**case 2:** $x$ is in a component of order at least 3

Then $x$ is in a component of order at least 6. It is enough to show that the average degree in this component is more than 2. Note that the only connected graph with average degree exactly 2 and precisely one vertex of degree 1 is a cycle with a pendant path. This is not $S_{3,3}$-saturated. \[\square\]

**Theorem 11.** Let $n \geq t^3$, $tn/2 \leq sat(n, S_{t+1,t+1}) \leq \frac{tn}{2} + \frac{t(t+2)}{2}$.
Proof. Let \( q = \lfloor n/(t+1) \rfloor \) and \( r = n - q(t+1) \). Then the graph \( G = (q - r)K_{t+1} \cup rK_{t+2} \) is \( S_{t,t} \)-saturated and \( |E(G)| \leq \frac{t(n-r)}{2} + r(t+2) \leq \frac{tn}{2} + \frac{t(t+1)}{2} \).

For the lower bound, observe that it is enough to show that every \( S_{t+1,t+1} \)-saturated graph has average degree at least \( t \). Assume there exists a \( S_{t+1,t+1} \)-saturated graph \( G \) such that \( \delta(G) < t \). Let \( x \in V(G) \) such that \( \deg(x) = \delta(G) = d \). Then all the vertices in \( V(G) \setminus N[x] \) (where \( N[x] \) denotes the closed neighborhood of \( G \)) must have degree at least \( t \) and must be adjacent to a vertex of degree at least \( t+1 \). Thus \( \sum_{i=1}^{n} \deg(v_i) \geq (d+1)d + t(n-d-1) + (n-d-1)/(t+1) = tn + \frac{n}{t+1} - (d+1)(t-d+\frac{1}{t+1}) \geq tn \) for \( n \geq t^3 \).

\[ \Box \]

**Theorem 12.** Given the double star \( S_{t+1,s+1} \) where \( t > s \) and \( n \) is sufficiently large, say \( n \geq 2s^2(t+1) + d + 1 \), then \( (\frac{s}{2}) n \) is \( \text{sat}(n, S_{t+1,s+1}) \) \( \leq \frac{s+1}{2}n - \frac{s^2+8}{8} \).

Proof. To obtain the upper bound, consider the graph \( G = K_1 + H \) where \( H \in \text{SAT}(n - 1, S_{s+1}) \). The graph \( G \) is \( S_{t+1,s+1} \)-saturated and \( |E(G)| = n - 1 + \frac{s+1}{2}n - \frac{s^2}{8} = \frac{s+1}{2}n - \frac{s^2+8}{8} \).

To obtain the lower bound, observe that in any \( S_{t+1,s+1} \)-saturated graph \( G \), vertices of degree \( s - 1 \) or less must be adjacent. Let \( x \in V(G) \) such that \( \deg(x) = \delta(G) = d \leq s - 1 \). Let \( y \in G - N[x] \). Then \( \deg(y) \geq s \) and if \( \deg(y) \leq t - 1 \), \( y \) must be adjacent to a vertex of degree at least \( t+1 \). Let \( r \) be the number of vertices in \( G \) with degree at least \( s \) and no more than \( t - 1 \).

If \( r \geq \frac{n-d-1}{2} \), then the degree sum of \( G \) is at least \( d(d+1) + (n-d-1)s + \frac{r}{t+1} \geq d(d+1) + (n-d-2)s + s^2(t+1) \geq ns \). If \( r \leq \frac{n-d-1}{2} \), then the degree sum of \( G \) is at least \( d(d+1) + (n-d-1)s + \frac{n-d-1}{2}(t-s) \geq ns \).

\[ \Box \]

**SUBDIVIDED STARS**

Let \( S^r_t \) denote the graph obtain by subdividing \( r \) edges of a star on \( t \) vertices. (See figure 5.)

![Figure 7: \( S^2_7 \)](image)

So \( S^r_t \) contains \( t + r \) vertices of which \( t - 1 \) have degree 1, \( r \) have degree 2, and one has degree \( t - 1 \). The vertex of degree \( t - 1 \) is called the center of the subdivided star. Recall that the tree on
vertices of minimum saturation number (previously referred to as \(T^*\)) can be thought of as \(S_{k-1}^1\).

We will find exact values for \(\text{sat}(n, S_r^2)\) for all \(r\) and establish upper and lower bounds for in the remaining cases.

**Theorem 13.** For \(n \geq 10\), \(\text{sat}(n, S_4^2) = n - \left\lfloor \frac{n}{10} \right\rfloor\).

**Proof.** Let \(P_6 = x_1x_2 \cdots x_6\) be a path on 6 vertices. Let \(H\) be the graph on 10 vertices constructed from \(P_6\) by adding precisely two pendant vertices to each of \(x_2\) and \(x_5\). Then the forest \(G = (\left\lfloor \frac{n}{10} \right\rfloor - 1)H \cup H^*\) where \(H^*\) is a copy of \(H\) with \(r \equiv n \mod 10\) additional pendant vertices added to \(x_2\). Then \(G\) is \(S_4^2\)-saturated.

We need to show that \(H\) is a \(S_4^2\)-saturated tree of smallest order. Let \(T\) be any \(S_4^2\)-saturated tree and let \(P = x_1x_2 \cdots x_m\) be a longest path in \(T\). Since neither a star nor a double star can be \(S_4^2\)-saturated, we know \(m \geq 5\). In \(T + x_1x_3\), vertex \(x_3\) must be the center of the newly obtained copy of \(S_4^2\) and vertex \(x_2\) must be adjacent to an additional pendant vertex. Adding the edge between these two vertices of degree 1, forces vertex \(x_2\) have have three pendant vertices. But this argument applies at the other end of the longest path. It is simple to check that if \(m = 5\) the graph is not \(S_4^2\)-saturated. So \(m = 6\) and \(H\) is the minimal tree. Observe that neither \(K_1\) nor \(K_2\) can be components and the result follows. \(\square\)

**Theorem 14.** If \(d \geq 4\) and \(n \geq d^2 - d + 1\), then \(\text{sat}(n, S_{d+1}^2) = n - \left\lfloor \frac{n+d-1}{d^2} \right\rfloor\).

**Proof.** Let \(H\) be a rooted tree with three levels such that the root \(r\) has \(d - 1\) neighbors and each of these has \(d\) children. Observe that \(H\) is \(S_{d+1}^2\)-saturated as is \(H \cup H\) and any graph constructed by adding additional pendant vertices to those in the middle level. Let \(H^*\) be the tree constructed from \(H\) by deleting \(d - 1\) pendant vertices from a single vertex. Thus, in \(H^*\) all vertices in the second level have \(d\) children except for one which has only one child. Note \(H^*\) is \(S_{d+1}^2\)-saturated as is any graph obtained by adding pendent vertices to those with \(d\) children. Note, \(H^* \cup H^*\) is not \(S_4^2\)-saturated. In fact no \(S_{d+1}^2\)-saturated graph can have two nonadjacent vertices each of degree 2. Thus, given any \(n \geq d^2 - d + 1\), there exists a \(S_{d+1}^2\)-saturated forest each component of which is a copy of \(H^*\) or \(H\), possibly with some additional pendant vertices.

We will now show that \(H^*\) is the smallest \(S_{d+1}^2\)-saturated tree and if \(T\) a tree such that \(H^* \cup T\) is \(S_{d+1}^2\)-saturated, then \(|V(T)| \geq |V(H)|\). Let \(T\) be an \(S_{d+1}^2\)-saturated tree. Since neither the star nor the double star are \(S_{d+1}^2\)-saturated, we know the longest path in \(T\) has at least 5 vertices. If
there exists a vertex \( x \) adjacent to two pendant vertices, then \( x \) must be the center of the copy of \( S^2_{d+1} \) obtained by adding the edge between these end vertices. Thus, \( x \) must be adjacent to at least \( d \) pendant vertices. So, any vertex adjacent to a pendant vertex is adjacent to exactly one pendant vertex or at least \( d \) pendant vertices. Thus, without loss of generality, if the longest path in \( T \) is labeled \( x_1x_2\cdots x_m \), we can assume \( x_2 \) is adjacent to at least \( d \) vertices of degree 1. Now \( x_3 \) must be the center of the copy of \( S^2_{d+1} \) formed by the addition of edge \( x_1x_3 \). Thus, \( x_3 \) has exactly \( d-3 \) neighbors off the path. If any one of the neighbors of \( x_3 \) is a pendant vertex, adding the edge between it and vertex \( x_2 \) will produce a contradiction. Finally, at most one of the \( d-1 \) neighbors of \( x_3 \) can have degree 2. Thus all but at most one must be adjacent to at least \( d \) pendant vertices. Thus all \( S^2_{d+1} \)-saturated trees have at least \( |V(H^*)| \) vertices and any longest path in such a tree shares the structure of \( H^* \) at one end. Since no \( S^2_{d+1} \)-saturated graph can have two nonadjacent vertices of degree two, the result follows.

Next we will consider the evenly subdivided star \( S^t_{t+1} \). (See figure 5.) Note that while \( S^t_{t+1} \) is a tree with many vertices of degree 2, there does not exist an \( S^t_{t+1} \)-saturated forest.

![Figure 8: \( S^5_6 \)](image)

**Theorem 15.** For \( t \geq 3 \) and \( n \geq 3t \), \( n \leq \text{sat}(n, S^t_{t+1}) \leq n + 3t - 5 \)

**Proof.** Let \( G = (K_1 + \{(t-1)K_3 \cup (n-3t)K_1\}) \cup K_2 \). The graph \( G \) is \( S^t_{t+1} \)-saturated and provides the upper bound.

We will prove the lower bound in steps. Let \( G \) be a \( S^t_{t+1} \) saturated graph.

**Claim 1:** If \( G \) has a vertex of degree 2, then its neighbors are adjacent.

Assume \( x \) is a vertex of \( G \) of degree two and its neighbors, \( y_1 \) and \( y_2 \) are nonadjacent. Then \( G + y_1y_2 \) contains a copy of \( S^t_{t+1} \) which must use edge \( y_1y_2 \) and at most one of the edges incident to \( x \). But this implies \( G \) itself contains a copy of \( S^t_{t+1} \), a contradiction.

**Claim 2:** If \( G \) has a vertex of degree 1 in a nontrivial component, its neighbor must lie on a cycle.
Assume $G$ has a vertex $x$ of degree 1. Let $y$ be its neighbor and assume $y$ lies on no cycle in $G$. Since a star is not $S_{t+1}^t$ saturated, there must exist a neighbor of $y$, say $z$, of degree 2 or more. Consider $G + x, z$. The vertex of degree $k$ in the newly created copy of $S_{t+1}^t$ is either $y, z$, or neighbor of $z$. But in each case, the copy of $S_{t+1}^t$ using edge $xz$ forces a copy of $S_{t+1}^t$ in $G$.

Thus, $G$ has no nontrivial components that are trees. In fact, no component of $G$ is unicyclic, since any such component would have to be a single cycle with attached pendant edges which is not $S_{t+1}^t$ saturated. So any nontrivial component $C$ must have at least $|V(C)| + 1$ edges. Thus, even if $G$ has a $K_1$ or $K_2$ as a component, the lower bound holds.

QUESTION: I think the upper bound is right. How can the argument for the lower bound be improved?

Define $S_t^r$ to be the graph obtained by subdividing $r$ edges of a star $S_t$ on $t$ vertices. So $S_t^r$ has $t + r$ vertices. (See figure 5.)

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{S6_2.png}
\caption{$S_{6,2}$}
\end{figure}

**Theorem 16.** For $r \geq 3$, $n \leq \text{sat}(n, S_t^r) \leq n + 3r - 5$

*Proof.* The upper bound is obtained by observing that the graph $G = K_1 + \{(r - 1)K_3 \cup (n - 3r + 2)K_1\} \cup K_2$ is $S_t^r$-saturated.

For the lower bound we argue by contradiction first that no nontrivial component of an $S_t^r$-saturated graph can be a tree and second, that no nontrivial component is unicyclic.

Assume that there exists a tree $T$ that is $S_t^r$-saturated. Now find a longest path, $P = x_1, x_2, \ldots, x_k$ in $T$. Then $k \geq 5$. If $x_2$ is adjacent to two end vertices, then adding the edge between them forces $x_2$ to be the center of a copy of $S_t^r$ which would again force $T$ to contain a path longer than $P$. Thus, $\deg(x_2) = 2$. Now consider $T + \{x_1x_3\}$. The vertex $x_3$ must be the center of the newly obtained copy of $S_t^r$ and the new edge must be used as a pendant edge in $S_t^r$. So $\deg(x_3) = t - 2$ and $T$ must contain exactly $r - 1$ additional paths of length exactly 2 starting at $x_3$ disjoint from the path $P$.  

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For each of these paths the vertex adjacent to $x_3$ has degree exactly 2. Label one such vertex $y$. Now $T + \{x_2y\}$ fails to contain a copy of $S'_i$. So there does not exist an $S'_i$-saturated tree.

Assume there exists a unicyclic $S'_i$-saturated graph $G$. Clearly, $G$ cannot simply be a cycle, so $G$ is a cycle with pendant trees. Let $x$ be a vertex whose distance from the cycle is maximized. If $x$ is a distance 1 away from the cycle, let $y$ be the neighbor of $x$ and let $y^-$ be a neighbor of $y$ on the cycle. Then $G + \{xy^-\}$ fails to produce a copy of $S'_i$. So $x$ is at least a distance 2 away. Now the argument used on trees will show no such graph is $S'_i$-saturated. Thus, if $G$ is a nontrivial $S'_i$-saturated graph, then $|E(G)| \geq |V(G)| + 1$ and the lower bound follows.

CATERPILLARS

Define a caterpillar, $P^d_k$, to be a path on $k \geq 5$ vertices such that all interior vertices of the path are adjacent to $d$ additional pendant vertices. (See figure 5.) So $|V(P^d_k)| = k + (k - 2)d$. Note that if $k = 4$, the graph would be a double broom. If $k = 3$, the graph would be a star.

![Figure 10: $P^3_5$](image)

**Theorem 17.** If $n \geq 2k - 2$, $k \geq 5$, then

$$n \leq \text{sat}(n, P^1_k) \leq (k - 3) \left( \frac{k - 2}{2} \left\lfloor \frac{n}{2k - 2} \right\rfloor \right).$$

**Proof.** Let $H = K_{k-3} + K_{k+1}$. Let $G = (\lfloor \frac{n}{2k-2} \rfloor - 1)H \cup (K_{k-3} + K_{k+r+1})$ where $r \equiv n \mod 2k-2$. Note $G$ is $P^1_k$-saturated. The lower bound follows from the observation that if a $P^1_k$-saturated graph has a vertex $x$ of degree 0 (or 1), then all $n - 1$ (or $n - 2$) vertices nonadjacent to $x$ have degree at least 2. Furthermore at least 3 vertices have degree 3 or more.

**Theorem 18.** For $n \geq k + (k - 2)d$, $k \geq 4$ and $d \geq 2$, $\frac{d+1}{2}n - \frac{(d+2)^2}{8} \leq \text{sat}(n, P^d_k) \leq \frac{2k+d-7}{2}n - \frac{(k-3)(k-2)}{2} - \frac{d^2}{8}$. 

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Proof. The lower bound follows directly from Theorem 5 and the fact that $\delta_2(P_d^k) = d + 2$. The upper bound follows from the observation $G = K_{k-3} + H$ where $H \in \text{SAT}(n-k+3, S_{d+1})$ is $P_d^k$-saturated. \hfill \Box

Assume that, given a $P_d^k$, we lengthen the path on $k$ vertices by adding a vertex at each end (See figure 5.) So this appended caterpillar (call it $AP_{k+2}^d$) has a path on $k + 2$ vertices such that the middle $k - 4$ vertices have degree $d + 2$ and it has two vertices of degree 2.

![Figure 11: $AP_7^3$](image)

**Theorem 19.** For the graph $AP_{k}^d$, $n \geq k + (k - 4)d$, $k \geq 6$ and $d \geq 2$, we have $n \leq \text{sat}(n, AP_{k}^d) \leq n(k-3) + (k - 3)\lfloor \frac{n}{k+(k-4)d} \rfloor ((k-4)d + 2)$.

**Proof.** Let $H = K_{k-3} + \overline{K}_{(k-4)d+3}$. Then the graph $G = \left(\left\lfloor \frac{n}{k+(k-4)d} \right\rfloor - 1 \right) H \cup (K_{k-3} + \overline{K}_{(k-4)d+3+r})$ where $r \equiv n \mod k + (k - 4)d$ is $AP_{k}^d$-saturated.

The lower bound can be obtained by observing no nontrivial component of an $AP_{k}^d$-saturated graph can be a tree or unicyclic. The longest path argument used in the proofs in the section on subdivided stars will apply. \hfill \Box

On the other hand, we know that if we append to $AP_{k}^d$ a path on $k$ vertices at one end to make a sort of "one-sided" caterpillar, then the saturation number drops below $n$.

**NOTE TO ME:** This lower bound should be in terms of $d$. **ALSO:** The case where $d = 1$ will have to be separate.

**REMINDER:** Go back and check how caterpillars and double brooms fit together.

**QUESTION:** What happens if you append caterpillars more? What happens if the caterpillar legs get longer?

**COLLECTION OF PATHS**

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Theorem 20. For $n \geq 6t - 6$, $\lceil \frac{n+3t+1}{2} \rceil \text{sat}(n, tP_3) = \lceil \frac{n+6(t-1)}{2} \rceil$.

Proof. The graph $N$ (sometimes called a net) consists of a $K_3$ such that each vertex of this triangle is adjacent to precisely one pendant vertex. So $N$ has 6 vertices and 6 edges. The graph $H_{n,t}$ is the graph on $n$ vertices consisting of $(t-1)$ copies of $N$ and $\lfloor \frac{n-6(t-1)}{2} \rfloor$ copies of $K_2$. If $n$ is odd, one component of $H_{n,t}$ is an isolated vertex. Then $H_{n,t-1}$ is $tP_3$-saturated.

We can assume $t \geq 2$ since the case $t = 1$ is known from [KT86]. Let $G$ be a minimally $tP_3$ saturated graph. Then $G$ can have at most one isolated vertex. Also, if $G$ contains a vertex of degree 2, its neighbors must be adjacent which follows from the observation that $P_3$ using this new edge can also be obtained without it and remain disjoint from the other $(t-1)$ copies of $P_3$.

Now, we can argue that $G$ has no nontrivial component that is a tree. Assume otherwise. Let $P = x_1, x_2, \cdots, x_k$ be a longest path in this $rP_3$-saturated tree. We know $2 \leq r \leq t$. Since a star is not $rP_3$-saturated, we know $k \geq 4$. Then $x_2$ must have at least one additional pendant vertex other than $x_1$, say $y$. But adding edge $x_1y$ cannot produce a new copy of $P_3$. So any component of $G$ of order at least 3 contains a cycle. In fact, no vertex in $G$ can be adjacent to two pendant vertices. Thus, any component $C$ that is unicyclic and $rP_3$-saturated, must be a cycle $C_l$ such that every vertex of this cycle has exactly one pendant vertex. (So $|V(C)| = 2l$.) By adding a smallest chord to the cycle, it follows that no such graph is $rP_3$-saturated. Thus, every nontrivial component of $G$ has at least two cycles. Thus a minimum of edges could be achieved by a single $tP_3$-saturated component and the remaining vertices in a matching. The number of edges in this graph is at least $3t + 1 + \lfloor \frac{n-3t}{2} \rfloor$. \hfill \Box

In fact I think the upper bound can be lowered by making a component consisting of triangles strung together.

QUESTION: For a tree of fixed order, follow the evolution of sat number through different trees. What is driving this? What is the second largest and second smallest sat number?

References


