Combinatorial Games
Contributed by: Math on my Mind

Purpose: to problem solve and think critically through playing games

Background

Tic-tac-toe is among the most common of childhood games. Although this game is often mastered at an early age, it takes detailed analysis to describe the game mathematically. For instance, there are 255,168 possible games that result in one of 138 board positions. Of these unique positions, the first player wins 91, the second player wins 44, and the game ties in 3.

Even if the second player plays tic-tac-toe perfectly, the first player will never lose if they also play perfectly. While children accept this to be true based on experience, combinatorial game theory proves this to be true.

Overview

This activity introduces students to combinatorial game theory by playing and analyzing some simple games.

Materials

Chalk board or similar instructional means
Pencil and paper for students

Chomp

Chomp is played on a square grid of ‘cookies’ of which the top left is ‘poisoned’. Two players take turns picking a cookie and ‘eating’ (removing) the rectangular grid of cookies to the right and below of the selected cookie. The player who eats the poisoned cookie loses.

After playing a few rounds at the board, encourage students to play a few rounds with a partner and find a winning strategy. A discussion of the strategy stealing argument to show that the first player can always win fits in well before letting students play on their own.

Test some of the students’ strategies before showing them a winning strategy for the first player. One such winning strategy begins with the first player picking the cookie immediately diagonal of the poisoned cookie. This leaves the first row and first column remaining. The first player then mirrors the second player’s move across the diagonal. For example, if the second player...
removes two (2) cookies from the first column, the first player will then remove two (2) cookies from the first row.

Surprisingly, there is no known winning strategy for playing chomp on a general rectangular grid!

**Subtraction Games**

Subtraction games can be played verbally or on a piece of paper between two players, say Players A and B. Player A starts by picking a starting number (we suggest starting around 20 for classroom activities), and then Player B decides who will subtract first. Then players take turns subtracting 1, 2, or 3. The player who subtracts to zero (0) wins. A player may not subtract past zero, so the game can be viewed as taking away items from a pile.

Similar to Chomp, play a few games on the board and then encourage students to determine a winning strategy while they play on their own. The following winning strategy can be deduced by examining the numbers from which a player can win and then subtracting so that the other player is not at one of those numbers. Specifically, the strategy is to subtract so that the opposing player is left with a multiple of four (4). Since Player A determines who goes first, Player A has the ability to determine who wins.

**Modifications and Extensions**

The \(m,n,k\)-game is played on an \(m\) by \(n\) grid by two players who take turns by claiming squares in an attempt to get \(k\) squares in a row. Tic-tac-toe is the 3,3,3-game. As in tic-tac-toe where the second player can not win if both players play optimally, a standard strategy stealing argument (see reference) can be used to show that the second player can not win the \(m,n,k\)-game if both players play optimally.

A strategy stealing argument (see reference) can be used to show the existence of a winning strategy for Chomp and Subtraction Games.

For subtraction games, students can investigate how the winning strategy changes when subtracting from different number sets such as \{1,2,3,4\}, \{1,3,5\}, or \{1,2,5\}, which are won by multiples of 5, 2, and 3, respectively.

**Some References**

http://www.se16.info/hgb/tictactoe.htm
http://www.mathrec.org/old/2002jan/solutions.html
http://plus.maths.org/content/mathematical-mysteries-chomp