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Interval Analysis, Fuzzy Set Theory and Possibility Theory in Optimization

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Abstract: This study explores the interconnections between interval analysis, fuzzy interval analysis, and interval, fuzzy, and possibility optimization. The key ideas considered are: (1) Fuzzy and possibility optimization are important methods in applied optimization problems, (2) Fuzzy sets and possibility distributions are distinct which in the context of optimization lead to flexible optimization on one hand and optimization under uncertainty on the other, (3) There are two ways to view possibility analysis leading to two distinct types of possibility optimization under uncertainty, single and dual distribution optimization, and (4) The semantic (meaning) of constraint set in the presence of fuzziness and possibility needs to be known a-priori in order to determine what the constraint set is and how to compute its elements. The thesis of this exposition is that optimization problems that intend to model real problems are very (most) often satisficing and epistemic and that the mathematical language of fuzzy sets and possibility theory are well-suited theories for stating and solving a wide classes of optimization problems.

Keywords: Fuzzy optimization, possibility optimization, interval analysis, constraint fuzzy interval arithmetic

1 Introduction

This exposition explores interval analysis, fuzzy interval analysis, fuzzy set theory, and possibility theory as they relate to optimization which is both a synthesis of the author's previous work bringing together, transforming, and updating many ideas found in [49],[50], [51], [52], [58], [63], [64], [118], [116], and [117] as well as some new material dealing with the computation of constraint sets. It is assumed that the reader is familiar with basic notions associated with fuzzy set theory, possibility theory, and optimization under uncertainty.

A real-valued optimization model, as is well-known, which is the focus of this study, is a ***normative*** mathematical model whose underlying system is most often ***constrainted***. Thus, from this perspective, there are two components to an optimization model:

1. ***The normative process*** - A normative process comes with or has imposed on it a performance measure which is an explicitly articulated criterion for system behavior. The "explicitly articulated criteria" is typically a real-valued mathematical function. Implicitly, "performance measure" requires a way of ordering the inputs (decisions which are the variables of the problem) and thus the requirement is that whatever the domain of the process, its range must be a complete ordered lattice on which the performance is measured. The set of real numbers is the usual complete ordered set on which the performance measures are carried out. Thus, real-valued functions are often used as "the explicitly articulated criteria" which transforms the domain of the problem to the real numbers. This real-valued function is called the *objective function*. The objective function must be computable on every element of the domain and so the domain must be endowed with a structure that allows this. Dynamic optimization typically calls the normative measurement criteria a *functional* which transforms the domain, a space of functions, into real-numbers, very often in the form of an integral.

2. ***The constraints*** - When deterministic mathematical formulations are used, constraints are mathematically described as a set of real-valued functional relationships whose solution is a set of real vectors called *the constraint set*. In the presence of fuzziness and/or possibility, what is a constraint set must be determined prior to computing its elements.

Objective functions and constraints of an optimization mathematical model play a key role in the way we will develop the relationships between interval, fuzzy, and possibility optimization methods. Because we are interested in mathematical models of intervals, fuzzy sets, and possibility entities, let us mention two categories, ***deterministic*** and $\boldsymbol{non-deterministic}$ mathematical models because this will put the models of interest to this presentation within one of two contexts. Deterministic models, for our purposes, are those all of whose aspects from initial statements of the model to the final solution(s) are real numbers and real relationships. Non-deterministic models are models that are not deterministic. This classification, deterministic and non-deterministic, puts the focus of this exposition as a subset of non-deterministic models. The types of models we are interested in can be illustrated by the following example.

Example 1 *(Radiation Therapy of Tumors - [53])) The determination of how to use particle beams to treat tumors is called the radiation therapy problem (RTP). Beams of particles, usually photons or electrons, are oriented at various angles and with varying intensities to deposit dose (energy/unit mass) to the tumor. The idea is to deposit just enough radiation to the tumor to kill all the tumor cells while sparing normal tissue. The process begins with the patient's computed tomography (CT) scan. Each CT image is examined to identify and contour the tumor and normal structures. Each image is subsequently vectorized. Likewise, candidate beams are discretized into beamlets, where each beamlet is the width of a CT pixel. A pixel is the mathematical entity or structure (a square in the two-dimensional case and a cube in three dimensions) that is used to represent a unit area or volume of the body at a particular location. For two-dimensional problems, about seventeen CT scans (slices) are sequentially "stacked" (to form a three-dimensional image that covers the tumor), and a variety of resolutions might be considered,* 256×256, *or* 512×512 *are typical resolutions. One set of beams each at ten or more equally spaced angles is not unusual. Since we constrain the dosage at each pixel, for ten equally spaced angles, the complexity of the problem ranges from an order of* $17\cdot10\cdot64^2$ *to* $17\cdot10\cdot512^2$ *potential constraints. However, since all pixels are not in the paths of the radiation beams that hit the tumor, and some are outside the body, we a-priori set the delivered dosages at these pixels to zero and remove them from our analysis. This corresponds to blocking the beam, which is always done in practice. The identification of a set of beam angles and weights that provide a lethal dose to the tumor cells, while sparing healthy tissue with a resulting dose distribution acceptable and approved by the attendant radiation oncologist, is called a treatment plan. The largest actual problem reported by [53] is on the order of* $500,000$

constraints in a fuzzy/possibility optimization problem. A dose transfer matrix A^T (representing how one unit of radiation intensity in each beamlet is deposited in pixels - for historical reasons, we use a transpose to emphasize its origin as the discrete version of the inverse Radon transform), called here the attenuation matrix, specific to the patient's geometry, is formed where columns of A^T correspond to the beamlets and rows represent pixels. A component of a column of the matrix A^T is non-zero if the corresponding beamlet intersects a pixel, in which case the value is the positive fraction of the area of the intersection of the pixels with the beamlet otherwise it is zero. The beams then are attenuated according to a factor dependent on the distance from where the beam enters the body to a pixel within the body and the type of tissue at that pixel. This attenuation is of the form

$$e^{-\mu d}$$

where d is the beam distance from the edge of the body to the pixel for which the dosage is being computed The variables are vectors x that represent the beamlet intensities. The deterministic mathematical programming problem is

$$\begin{aligned} z &= \min f(c,x) \\ A^T x &\leq b, \end{aligned}$$

where an example of a normative criterion, $f(c,x)$, might be for this context, minimization of total radiation. This problem has four aspects which transform it into a fuzzy/possibility optimization problem. First, the objective function can be a probability function (upper/lower bounding), the probability that the radiation intensity vector x will turn a health pixel into a cancerous one. Each row of the left side, that is, each row of the constraint matrix A^T, represents each pixel in the path of the beam, the "beam's eye view" of the tumor. Thus, each row accumulates pixel by pixel total radiation deposited by the radiation intensity vector x at that pixel. This will occur mathematically since the dot product of the i^{th} row vector (the i^{th} pixel) A_i^T with the (column) vector x is the sum of radiation at the pixel, $A_i^T x$. Since a pixel can be cancerous, non-cancerous, or cancerous to a degree (the boundaries between cancerous and non-cancerous are gradual, transitional and thus fuzzy), the (some or all) components of the left side matrix A^T matrix may be composed of fuzzy intervals. The right-hand side value is the maximum allowable dosage (for critical organs in the path of the beam and maximal *value representing pre-burning for the tumor cells). Separately, for tumor cells, there is also an associated minimal value, the smallest value a radiation oncologist does not wish to allow the radiation to go below (it is the minimal acceptable radiation that will kill a cancer cell). These values may be considered to be possibility since these are values derived from research, expert knowledge (epistemic values), and experience with preferred values within the range. Therefore, the right-hand side value may be considered either as a target (with preferred ranges) or as a purely possibility value. The objective function is often the minimization of the probability of the delivered dosage will turn a health cell cancerous. However, two (and more) other criteria may also be used: (1) The minimization of the maximum delivered dosage to the tumor, (2) The minimization of total radiation delivered by the linear accelerator.*

Remark 1 *The normative criteria, minimization of the probability that a healthy cell becomes cancerous, is described by a single real-valued probability distribution function. The constraints for the RPT are typically a mixture of fuzzy and possibility intervals.*

The general optimization problem for this presentation is

$$
\begin{aligned}
z &= \min f(x,c) && (1)\\
\text{subject to } \; g_i(x,a) &\le b \quad i=1,...,M_1 && (2)\\
h_j(x,d) &= e \quad j=1,...,M_2. && (3)
\end{aligned}
$$

The constraint set is denoted $\Omega = \{x \mid g_i(x,a) \le b \;\; i=1,...,M_1, h_j(x,d)=e,$ $j=1,...,M_2\}$. It is assumed that $\Omega \neq \emptyset$. The values of a, b, c, d and e are inputs (data), parameters of the programming problem. Given our interest, we restrict these parameters in our discussion to be intervals, fuzzy intervals or possibility distributions which we defined below. Moreover, the operator min and relationships $=$ and $\le$ can take on flexible or fuzzy meaning becoming "soft" operations, relationships, or constraints. For example, the equality and inequality relationships may be aspirations, that is, they may take on the meaning, "Come as close as possible to satisfying the relationships with some degree of violation being permissible." Moreover, "min" might mean a satisfactory cost. Of course, if the values of a, b, d, and/or e are interval, fuzzy interval, or possibility distribution, the meaning of the inequality must be specified as will be discussed. It is noted that when the objective function and/or constraints parameters are not real numbers, the optimization problem may not be (undoubted is not) convex in the classical sense even when the real-valued equivalent problem is. Thus, the usual solution methods are local even if the deterministic equivalent model is convex with local solutions being global. For example, in very simple cases where the constraint is linear, of the form $Ax \leq b$, and the coefficients of the matrix are intervals, the solution set can be a star-shaped region even for two variables and two constraints (see [29]). When the parameters a, b, d, and/or e are interval, fuzzy, or possibility, the underlying model is not known exactly or it may be that the model is precise but knowledge of what the values of the data are incomplete or deficient in some sense. The radiation therapy problem given above is an example of a fuzzy and/or possibility optimization.

There is another point of view in which the optimization problem statement begins as a fuzzy optimization statement rather than a real-valued optimization model some or all of whose parameters are fuzzy and/or intervals and/or possibility (see [84], [85]). For the purposes of this exposition, we begin with (1),(2) and (3). This study restricts itself to real-valued interval, fuzzy interval, possibility coefficients, and to soft constraints.

2 What Does Interval, Fuzzy, and Possibility Optimization Add to Our Understanding and Ability to Solve a Broader Class of Problems?

We make a case for fuzzy and possibility optimization by looking at the basics of optimization modeling and their the interconnections to interval analysis, fuzzy interval analysis, and interval and fuzzy/possibility optimization. Two key ideas are:

- Optimization models of real systems are almost always ***satisficing*** (see [102], [103]) in which case fuzzy and possibility approaches are key to these optimization models. Satisficing is defined by Herbert Simon (see [102], [103]) to means that decision makers rarely work with deterministically "the best" solution or even be able to obtain "the best" solution to a real problem, but seek to obtain a solutions are satisfying. Clearly, the usual deterministic models, if used to model decision processes described by Simon, must be modified. Fuzzy and possibility optimization are abel to model satisficing in a natural and direct way.

- Many satisficing optimization models are ***epistemic***. That is, models that are epistemic are those which we, as humans, construct from knowledge about a system rather than models that are constructed from the system itself. For example, an automatic pilot of an airplane models the system physics. A fuzzy logic chip that controls a rice cooker is a model of what we know about cooking rice rather than the physics of rice cooking.

Fuzzy and possibility optimization models are well-suited and a flexible approaches for representing satisficing and epistemic normative processes or problems. One begins any modeling process by stating the problem in its native setting. Since it is our thesis that many problems in optimization are flexible, satisficing, and/or epistemic as opposed to deterministic, the natural mathematical language in which these types of problems may be stated is fuzzy set theory and possibility theory.

Rommelfanger, H. J. [91] (page 295) states that the only operations research method that is widely applied is linear programming. He goes on to state that even though this is true, of the 167 production (linear) programming systems investigated and surveyed by Fandel (see [23]) only 13 of these were "pure" deterministic linear programs. Thus, Rommelfanger concludes that even with this most highly used and applied operations research method, there is a discrepancy between classical deterministic linear programming and what is actually done.

Deterministic and stochastic optimization models require well-defined input parameters (coefficients, right-hand side values), relationships (inequalities, equalities), and preferences (real valued functions to maximize, minimize) either as real numbers or real valued distribution functions. Any large scale model requires significant data gathering efforts. If the model has projections of future values, it is clear that real numbers and real valued distributions are inadequate representations of parameters, even assuming that the model correctly captures the underlying system. It is also known from mathematical programming theory that only a few of the variables and constraints are necessary to describe an optimal solution (basic variables and active constraints), assuming a correct deterministic normative criterion (objective function). Thus, only a few parameters need to be obtained precisely, those that are in the final basis. Of course the problem is that it is not known a-priori which variables will be basic and which constraints will be active.

Herbert Simon (see [102], [103]) states:

> "Of course the decision that is optimal in the simplified model will seldom be optimal in the real world. The decision maker has a choice between optimal decisions for an imaginary simplified world or decisions that are 'good enough,' that satisfice, for a world approximating the complex real one more closely. ... What a person cannot do he will not do, no matter how much he wants to do it. Normative economics has shown that exact solutions to the larger optimization problems of the real world are simply not within reach or sight. In the face of this complexity the real-world business firm turns to procedures that find good enough answer to questions whose best answers are unknowable. Thus normative microeconomics, by showing real-world optimization to be impossible, demonstrates that economic man is in fact a satisficer, a person who accepts 'good enough' alternatives, not because he prefers less to more but because he has no choice."

An email discussion with Professor Rommelfanger [92] relates the following.

> "In fact Herbert Simon develops a decision making approach which he calls *the concept of bounded rationality.* He formulated the following two theses. *Thesis* 1: In general a human being does not strive for optimal decisions, but he tends to choose a course of action that meets minimum standards for satisfaction. The reason for this is that truly rational research can never be completed. *Thesis 2*: Courses of alternative actions and consequences are in general not known a-priori, but they must be found by means of a search procedure."

A model of an actual problem is always an abbreviated view of the underlying actual system. If a problem were able to be manipulated in situ to obtain a solution without a symbolic representation, then there would be no need for modeling the problem mathematically. Inherently, a mathematical model is a symbolic representation of the problem not the problem itself. A corollary of this fact is: "If we have a problem, we don't know its solution. If we knew the solution, we would not have a problem. Since we don't know (the solution) at the point we have a problem, we uncover solutions in process." Thus, a model is the process of becoming. Mathematics is the science of precision. At the heart of analytical mathematics is order. Measure (quantity and its generalizations) and extent (area, integration) are derived from order. Therefore, the very essence of optimization (the normative) is order (what is the largest, what is the smallest). Order determines the measure we use (the objective function), and it is the objective function that tells us what is "best" for the context of the problem. A most ideal order is found in the set of real numbers (the real number line). In optimization we very often map (via the objective function) to the real numbers. In fuzzy sets we map onto the set of real numbers using aggregation operators. In possibility optimization as we define it for this exposition, we map possibility and necessity distributions to real numbers using an evaluation (utility) function whose domain is that of the set of distributions. For example, a generalized expectation is one such evaluation (utility) function.

However, Professor Roman Slowinsky quotes, in articulating a differing point of view that the real numbers are an ideal order:

> "Si l'ordre apparaît quelque part dans la qualité, pourquoi chercherions-nous à passer par l'intermédiaire du numbre?" Bachelard 1934 -
> "If an order appears somewhere in quality, why should we like to interpret this order through numerical values?"

A very useful approach to fuzzy and possibility optimization should also adhere to the *Principle of Least Commitment* which states that, "Only commit when you must " The principle of least commitment in the context of optimization is to only use a real-valued function when you must, preferably only at the "end" of the analysis. A corollary to this principle is, "Carry the full extent of uncertainty and gradualness until one must choose." This is the philosophy of the recourse model in stochastic optimization..

We summarize what has been articulated using, additionally, insights from [11].

- Fuzzy optimization(what we call here *flexible programming*), offers a bridge between numerical (deterministic) approaches and the linguistic or qualitative ones. The thrust of these approaches are to provide the analyst with what is the uncertainty in the results of a decision process.

- Fuzzy set theory and its mathematical environment of aggregation operators (*and*, *t-norms* - see the discussion below), interval analysis, constrained interval analysis (developed subsequently, [50], [52]), fuzzy interval analysis, constraint fuzzy interval analysis (developed subsequently, [50], [52]), gradual numbers (see [27], [118], [107]), and preference modeling (what we call flexible optimization and possibility optimition below), provide a general framework for posing decision problems in a more open way and provides a unification of existing techniques and theories.

- Fuzzy set theory has he capacity of translating linguistic variables into quantitative terms in a flexible and useful way.

- Possibility theory explicitly accounts for lack of information, avoiding the use of unique, often uniform, probability distributions.

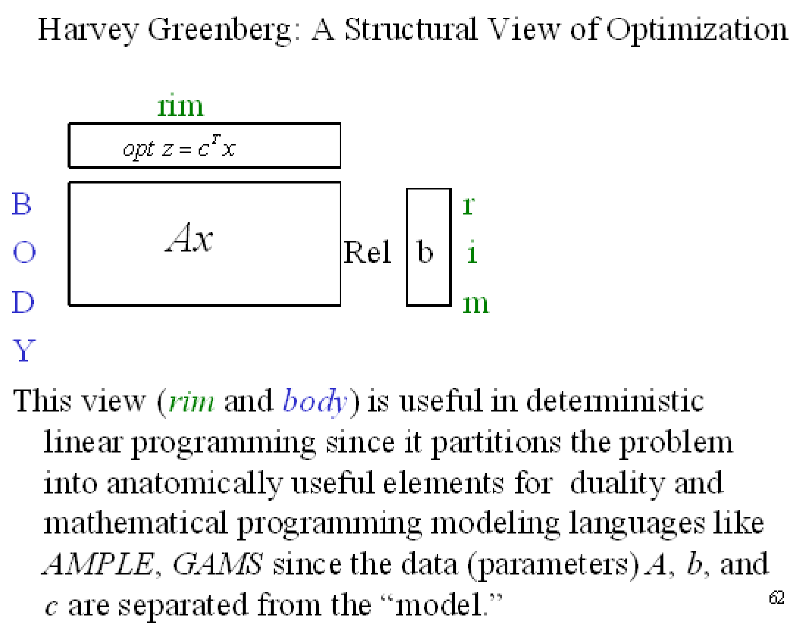
- The set theoretic view of functions to represent numbers (intervals) on which utilities are expressed as fuzzy sets offer a wide range of aggregation operations (t-norms).

In short, fuzzy set theory and possibility theory offer optimization an approach that comes close to the underlying flexible, satisficing, and information deficient processes that are the typical environment in which decision makers find themselves.

3 A Taxonomy of Fuzzy and Possibility Optimization for Our Generic Problem

The structure of the generic optimization problem (1),(2) and (3) may be considered to be formed by (from [30]) (i) the ***rim*** $f(x,c)$, for example, $c^T x$; and b, e, (ii) the ***body*** $f(x,a), h(x,d)$, for example, Ax; and (iii) the ***relationships***, $\leq, =$.

For the generic form of the mathematical programming problem (1),(2) and (3), we consider a taxonomy based on (i) rim objective function parameters, c, (ii) rim right-hand side parameters, b and e, (iii) body parameters a and



d, and (iv) relationship $\leq, =$. For this exposition, a fuzzy/possibility optimization problem is considered to be (1), (2) and (3) in the presence of data $\{a, b, c, d, e, \leq, =\}$ that is either all or a mixture of real, interval, probabilistic, fuzzy, possibility with at least one parameter being fuzzy or possibility. The types of optimization in the presence of interval, fuzzy interval, possibility coefficients, and soft constraints may be considered to be the following:

1. **Flexible Optimization**

   (a) **Soft constraints relationships**: The mathematical relationships $\leq$ and/or $=$ as "soft constraints" take on a flexible meaning. For example, we may have a soft less than or equal to constraint "I wish to be less than a certain value but I can tolerate going over the value beyond what I wish but not more." The extent of the flexibility will need to be specified. For our RTP example, a constraint on a critical organ in the path of the radiation beam may be, "I wish to be under 20 units of radiation for the spinal chord, but it must never exceed 30 units of radiation." Another type of flexibility is specification of a probability, possibility, or necessity of a constraint violation. These are chance constraints if probability and modal values in the case of possibility/necessity (see [37]).

(b) **The objective function expresses an acceptable set of desired targets**: For example, if an ideal amount of profit that is only attained under a rare confluence of situations is known (it happened only once before), then an object would be to come as close to that as possible (or exceed it). For the RTP objective function, a target might be, "Try to stay below R units of total radiation delivered to the lungs." These models include the original Bellman and Zadeh model [1] and Zimmermann's model [131].

(c) **The right-hand-side value of a constraint is a fuzzy interval which is semantically a target:** For example, for the RTP example, we might have, the constraint, "Do not deliver less than 58 units of radiation to the tumor, preferably between 59 units and 61 units, but never exceed 62 units." This is depicted in Figure 4.When the fuzzy interval is semantically a target, we have a flexible constraint resulting is a flexible optimization problem. Such a right-hand side value is essentially a flexible relationship that is described by the fuzzy interval.

2. **Single Distribution Optimization**

(a) **Interval, fuzzy interval, possibility cost coefficients** of the objective function rim parameter c with real valued coefficient constraint coefficient $a, b, d, e \in \mathbb{R}$. In this case we have a *family* of objective functions and we need to minimize of this family of functions which is typically done using a utility function.

(b) **The objective function rim parameter $c \in \mathbb{R}$ with interval, probability, possibility, fuzzy interval, and one or two of the following** - body parameters interval, fuzzy interval, possibility a, d and/or rim right-hand side values b, e are possibility.

(c) **The right-hand side values are semanticall information deficiency**, lack of specificity. In this case, a given decision variable x would generate a possibility distribution.

3. **Dual Distribution Optimization**: Dual distribution optimization is used when one wants bounds on the risk of taking a particular action. These methods give upper (optimistic), lower (pessimistic), a minimal maximal regret.

   (a) When both possibility and necessity distributions are used in the model, then a minimization of maximum regret approach is arguably an excellent approach to choose a course of action (see [42], [44], [54], [59], [60], [116], [117]).

   (b) When one uses belief/plausibility distributions over nested focal elements, it is possible to create possibility/necessity pairs that are then used to compute upper and lower bounds on decisions based on the uncertainty of the input data or model itself. One is also able to obtain the minimal maximum regret. One can also begin with random sets and general upper/lower min/max solutions (see [116], [117]) to create possbility/necessity distributions.

4. **Mixed (Flexible, Single Distribution and/or Dual Distribution) Optimization**

   (a) **Interval-Valued Probability Measure Optimization**: Any of the coefficients a, b, c, d, e may be interval, fuzzy, possibility where there may be a mixture of types within one constraint statement. In this case, we may transform them into interval-valued probability distributions (see [59], [60], [117]).

   (b) **Random Set Optimization** - Any of the coefficients a, b, c, d, e may be interval, fuzzy, possibility where there may be a mixture of types within one constraint statement may be transformed into random sets (see [17],[116]).

One also might classify fuzzy and possibility programming according to whether or not the solution is a real valued fuzzy interval vector or a real valued vector. Possibility programming methods of Buckley [4] and his colleagues, Delgado [7] and his colleagues, and Verdegay [119] have considered fuzzy interval solutions. The methods to obtain fuzzy interval solutions are different than those that obtain a real valued solution. Nevertheless, they fall under possibility programming or random set programming of the above taxonomy.

4 Fundamental Entities of Fuzzy and Possibility Analysis

We next outline the relationship between intervals, fuzzy intervals, and possibility distributions before looking at interval analysis as it relates to fuzzy interval analysis and possibility theory applied to optimization. We begin by defining terms that will be of use.

Definition 1 ***Uncertainty:*** *Uncertainty is the state of not knowing the exact value or truth of an entity or proposition/statement. For example, there is uncertainty in "the" minimum radiation dosage that will kill a cancer cell in a particular pixel of a particular person's CT scan having a particular type of cancer.*

Definition 2 ***Fuzzy set:*** *A fuzzy set is a set whose elements possess a degree of belonging to the set. For example, a pixel in a CT scan containing a tumor is* ***both*** *cancerous to a certain degree* ***and*** *(conjunctive) non-cancerous to a certain degree. Fuzzy sets have the characterization that an entity is more than one thing at the same time to a certain degree. Another example in the RTP context is the location of a pixel within a particular breathing or breath holding patient. The location a pixel in a patient at time of radiation that corresponds to the same pixel on the CT scan is transitional.*

Definition 3 ***Randomness****: Randomness is due to variability and/or frequency of outcomes of a given entity. An example of randomness due to variability is rainfall at a particular location on the 9th of November. The paradigm of randomness whose measure is frequency is the number of pips that show on the top face of a die after it is rolled.*

Definition 4 ***Possibility:*** *Possibility is the lack of information about the value of an entity or truth of a proposition or statement. For example, the age of the outgoing president of Brazil. While the age is a* ***real number****, for me, it is one of many possible values (either ...* ***or*** *49* ***or*** *50* ***or*** *... 60* ***or*** *...). It is* ***disjuntive*** *(an* ***or****). A disjunctive description of an ill-known quantity x (for example the real number which is the age of the outgoing president of Brazil) is represented (for me for the example of outgoing President Lula's age) by a set of mutually exclusive elements (say a_α or a_β or a_γ or ...). This requires the set representation of possibility to be decomposable.*

Fuzzy sets do not have this disjunctive characteristic, they have a conjunctive characterization.

Remark 2 ***Fuzziness is not uncertainty.*** *Fuzzy sets are sets that model transitional set belonging. Fuzzy sets are conjunctive (they describe states of nature in which simultaneous belonging occurs such as in image segmentation). Uncertainty is a state in which the information at hand does not allow the determination of the truth or value of the entity under discussion. Uncertainty thus encompasses both variability (such as temperature), frequency information associated with randomness (such as the roll of a die), and information deficiency about an existent entity (my knowledge of the age of the outgoing president of Brazil). Variability and frequency are modeled by probability distributions. Variability and frequency requires precise variable information of all possible states and a single single-valued distributions (real functions). The domain values (all states) are precisely known (for example, rainfall or the pips on a die) described by a* ***single*** *(distribution) function, its probability density function. Possibility theory, on the other hand, requires a pair (possibility and necessity) of functions to have a more complete description of the uncertainty under consideration.*

Remark 3 ***Waiting for Professor Marcello Anile under the statue of Giuseppe Garibaldi downtown Catania, Sicily, Italy.*** *The appointment was for 1pm. I was to meet Professor Anile at the park where the statue of Garibaldi was located in central Catania at the confluence of many old narrow streets which opened on to a central expanse of road. Being an American, I arrived at the statue a little before 1pm. The lunch time traffic was extreme with scooters at times on the sidewalk, cars everywhere honking, the traffic lights were not working and even if they were it would have made little difference. I looked at this scene of what seemed to me chaos and utter confusion. It was 1:30pm and I was still under the statue of Garibaldi. At 1:45pm I looked at the traffic, since Marcello still had not come, and all of a sudden I saw this incredible dance of autos, scooters, and pedestrians. There was much shouting and bustle of activity, yet no rancor. As I kept looking, I continued to see this incredible choreographed dance of autos, scooters, and pedestrians.. At 2pm, I thought that it was time to call the university to see about Marcello and as I was about to give up when at 2:10pm, Marcello arrived. For this example, one can say that the traffic flow was fuzzy since it was both a chaotic, confusion,* ***and*** *a choreography that only Italians know* *how to dance in this way. There was no uncertainty in the traffic flow. My estimation of when Marcello would be at the statue was a possibility distribution where I used the measure of the fact that Marcello had gotten his Ph.D. at Oxford University and was Italian and I, an American guest. I knew that there is Japanese time, there is German time, there is Latin American time, and there is Italian time. I moderated my distribution based on the epistemic knowledge of Marcello to develop a possibility distribution. There is uncertainty associated with my possibility distribution of Marcello's arrival at the statue of Garibaldi. When he arrived he said, "Italians have not yet learned that automobiles are incompressible fluids." So, while my distribution was correct, my estimation of the effect of traffic on Marcello's arrival was not since my epistemic knowledge of downtown Catania traffic was deficient, as was, apparently, Marcello's.*

Remark 4 *When we talk about uncertainty, we will restrict ourselves to entities that are intervals, probability distribution types (single distributions, interval-valued probability, belief/plausibility, random sets), or possibility. When probability means "flexibility" in the context of our study of optimization, for example, as it is in chance constraints, we will mean to have these models fall into the class of flexible analyses.*

The following simple example shows that *probability alone* is insufficient to describe uncertainty of every type. Suppose all that is known is that

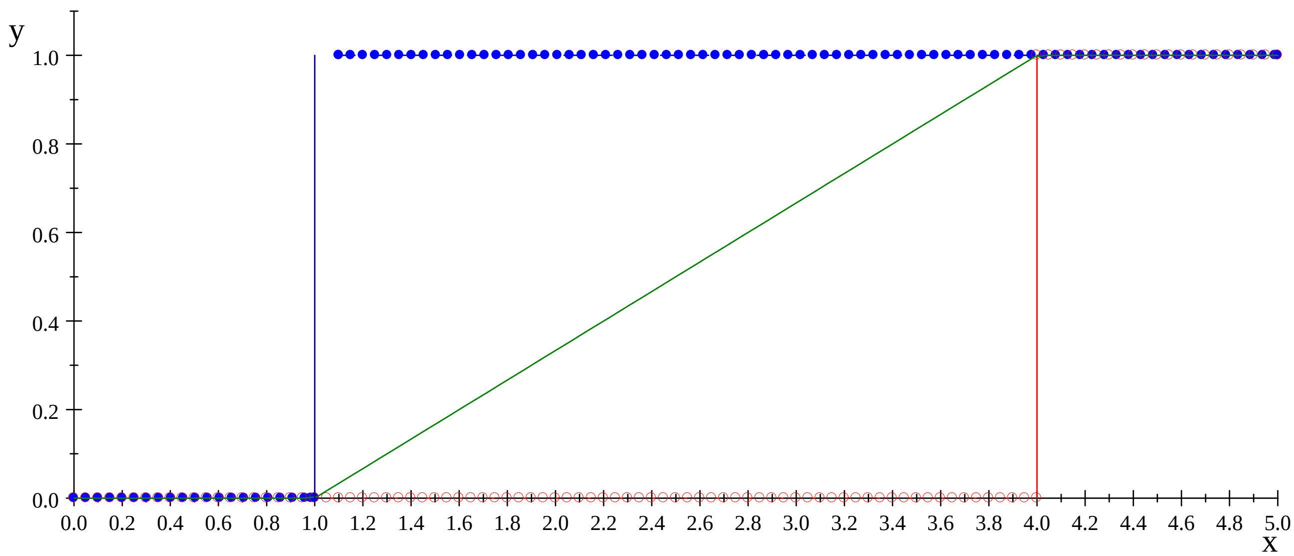
$$x \in [1,4]. \tag{4}$$

Very often such an interval describes what is actually known. For example, a reading coming from an instrument such as a thermometer is interval valued. Clearly, $x \in [1,4]$ implies that the real value that x represents is not certain, albeit bounded. If the uncertainty that $x \in [1,4]$ represents were probabilistic (x is a random variable that lies in this interval), then every distribution having support contained in $[1,4]$ would be equally valid given (4). Thus, if one chooses the uniform probability density distribution on $[1,4]$,

$$p(x) = \begin{cases} \frac{1}{3} & 1 \le x \le 4 \\ 0 & otherwise, \end{cases}$$

which is the usual choice given no other indication other than the support, one gives up information. The approach that keeps the entire uncertainty of (4) in adherence to the *Principle of Least Commitment,* considers $[1,4]$ as all distributions whose support is $[1,4]$. For this we no longer have *one* single cumulative density function but *two* that describe the bounds on all possible cumulative distributions. This is like an interval whose uncertainty can be described by *two* values (the left endpoint and the right endpoint). The pair of cumulative distributions that bound **all** cumulative distributions with support $[1,4]$ is depicted in Figure 1.

The statement $x \in [1,4]$ not only represents a random variable whose support is $[1,4]$, but it can be regarded as a mathematical entity, an interval, $[x] = [1,4]$.and as such is a complete, precise, and coherent entity in contradistinction with $[1,4]$ containing all probability density functions whose support is contained in this interval. The same object, $[1,4]$, may be interpreted as a probability and as an interval. We have at least two semantically distinct and analytically distinct meanings associated with the interval $[1,4]$ which have a different analysis, and metric.



Bounding Cumulative Distributions - Possibility (solid dots - blue), Necessity (open dots - red), Uniform (green)

For the semantic $x \in [1,4]$ being all probability functions whose support is in $[1,4]$, the upper cumulative distribution depicted in Figure 1 is a ***possibility distribution*** (solid dots - blue), and the lower cumulative distribution is a

necessity distribution (open dots - red). Therefore, when the statement $x \in [1,4]$ represents an unknown random variable whose support is the interval, to keep all the information about the uncertainty not only requires a pair of bounding functions, but a different arithmetic and mathematical analysis than straight forward functional analysis on probability distributions. The uniform distribution is precisely the intuitive solution to lack of information, "Choose the midpoint of the distribution pair as the solution *if one has to choose*." The cumulative distribution of the uniform distribution is the diagonal in Figure 1. Of course, the case is made here that $x \in [1,4]$ can also be tied to uncertainty which is purely non-probabilistic information deficiency in addition to an uncountably infinite set of random variables (whose support is contained in this interval). Why choose *a-priori* the uniform distribution for $[1,4]$ when one can carry the full uncertainty represented by $[1,4]$ as a possibility/necessity pair?

4.1 Intervals

An interval is a connected set of real numbers $X = [\underline{x}, \overline{x}] = \{x | \underline{x} \leq x \leq \overline{x}\}$. There are approaches that allow one or both endpoints to be infinite defined on the extended real numbers, but we do not pursue this generalization. Moreover, there is another generalization which allows the left endpoint to be larger than the right endpoint yielding what is called modal or improper intervals (see [28] or [86]). Modal or improper intervals can be used as intermediate results in computing with intervals but this is not developed here.

There is no "fuzziness" (transition) in an interval though an interval may be considered as a type of fuzzy interval (just as a real number may be considered as a type of complex number). Either an element belongs to an interval or it does not and thus has no transition from belonging to not belonging in contradistinction with a typical fuzzy set. An interval possesses a dual nature, that of a *"new" type of number* $[x] = \{\underline{x}, \overline{x}\}$ consisting of two elements (the lower bound and the upper bound) or as a *set* $[x] = \{x | \underline{x} \leq x \leq \overline{x}\}$. We will exploit this dual nature subsequently. Intervals may model non-specificity or information deficiency and are thus, for this setting, possibility, not fuzzy. However, intervals may be represented as a fuzzy interval (number) with a membership function. For example, the interval

$[1,4]$ has a fuzzy membership function

$$\mu_{[1,4]} = \begin{cases} 1 & x \in [1,4] \\ 0 & \text{otherwise} \end{cases}$$

The left/right bounds of the interval are vertical indicating a discontinuous, abrupt (lack of) transition. A general fuzzy entity is a transitional set represented by a "sloped" left/right part of the fuzzy membership function. We will discuss this at length subsequently. An interval belongs to uncertainty theory when the interval models non-specificity or lack of (complete) information.

Interval linear systems of equations problems for which we have $[A]x \leq [b]$ and $[A]x \geq [b]$ does **not** imply $[A]x = [b]$ unlike the real-value counterpart of the statement as can be seen in the following example.

Example 2 *Suppose that we wish to find the set of* $x \in \mathbb{R}$ *for which*

$$[2,3]x \leq [3,6] \text{ and} \tag{5}$$

$$[2,3]x \geq [3,6]. \tag{6}$$

The solution of (5) is, from a mathematical (precise) point of view,

$$x = (-\infty, 1].$$

The solution of (6) is

$$x = [3, \infty).$$

Thus (5) and (6) imply that $x = \varnothing$. *However,*

$$[2,3][\underline{x}, \overline{x}] = [3,6]$$

means that $[\underline{x}, \overline{x}] = [\frac{3}{2}, 2]$.

This example shows that for the constrained fuzzy/possibility linear system, $Ax \leq b$ and $Ax \geq b$ are not equivalent to $Ax = b$.

The foundations of interval analysis has a relatively long history (see [5], [22], [111], [122], [126], [69], [71]). We will use Moore's [71] approach and its extensions.

4.2 Fuzzy Intervals

What has been known as fuzzy number we will call fuzzy interval since a fuzzy interval is more general than what is called fuzzy number. It is well-know by this time that fuzzy sets capture transitional properties of entities both in the *abstract* (a gradual membership function rather than a Boolean one as in "smart" person) and in *reality* (the "line" separating tumor cells from non-tumor cells). Fuzzy sets are *not* uncertain in meaning nor in semantics. They are transitional or gradual. Thus, what we call here fuzzy optimization models normative processes possessing gradual or transitional properties of entities. Transition or gradualness in optimization is typically tied to *flexibility* in attainment of objectives or *flexibility* in constraint equations or inequalities "violations." Gradualness or transition is mathematically represented by the membership function and since the membership functions are of the coefficients or parameters of the optimization problem (1),(2) and (3), they will be in our usage, fuzzy intervals.

A fuzzy interval can be automatically translated into a possibility distribution, as we will see, and thus may be a model for both the lack of specificity as well as transition. A fuzzy interval is a way of modeling both gradualness and uncertainty/lack of specificity in optimization problems. That is, one way this (dual nature) occurs is when the entity (variable) is decomposable into mutually exclusive elements. Coefficients or parameters that are fuzzy intervals are by their very nature decomposable into mutually exclusive elements and thus possess a dual nature (gradual and uncertain). Fuzzy intervals are possibility distributions. Thus, fuzzy sets *may* have or take on a dual nature - that of capturing gradualness (flexibility) of belonging and capturing non-specificity, lack of information (uncertainty). A fuzzy set that is used in modeling uncertainty must be translatable into a possibility distribution given that fuzziness is not uncertain but transition. Possibility is tied to uncertainty. Flexibility and uncertainty are distinct. Fuzzy optimization and possibility optimization are distinct in both semantics and in solution methods as we will see subsequently though both use fuzzy interval representations.

4.3 Possibility Intervals

There seems to be a wider understanding of what fuzzy sets are both analytically and semantically than possibility theory. Therefore, we devote some effort to delineate quantitative possibility theory as we use it in optimization.

Possibility models non-specificity, information deficiency. The mathematical structure of possibility theory was first developed by [127] and more extensively articulate in [16], [19]. Since possibility is not additive but sub-additive, a dual to possibility, necessity, is required in order to have a more complete mathematical structure. Necessity was developed by [16]. In particular, if we know the possibility of a set A, it is not known what the possibility of the complement of A, A^C,in contradistinction to probability. The dual to possibility, necessity is required. Given the possibility of a set A, the necessity of A^C is known, but not the necessity of A.

> (From [13])"Limited (minimal) specificity can be modeled in a natural way by possibility theory. The mathematical structure of possibility theory equips fuzzy sets with set functions, conditioning tools, notions of independence/dependence, decision-making capabilities [lattices]. Lack [deficiency] of information or lack of specificity means we do not have "the negation of a proposition is improbable if and only if the proposition is probable." In the setting of lack of specificity, "the negation of a proposition is impossible if and only if the proposition is necessarily true." Hence, in possibility theory pairs of possibility and necessity are used to capture the notions of plausibility [possibility] and certainty [necessity]. When pairs of functions are used we may be able to capture or model lack of information. *A membership function is a possibility only when the domain of a fuzzy set is decomposable into mutually exclusive elements.* A second difference [between probability and possibility besides possessing a dual necessity] lies in the underlying assumption regarding a probability distribution; namely, all values of positive probability are mutually exclusive. A fuzzy set is a conjunction of elements. For instance, in image processing, imprecise regions are often modeled by fuzzy sets. However, the pixels in the region are not mutually exclusive although they do not overlap. Namely the region contains several pixels, not a single unknown one. *When the assumption of mutual exclusion of elements of a fuzzy set is explicitly made, then, and only then, the membership function is interpreted as a possibility distribution; this is the case of fuzzy intervals describing the ill-located unique value of a parameter*."

(the *italics* are my emphasis)

Moreover, possibility is always normalized since the semantics of possibility is tied to an existential entity. That is, models that use possibility are of existential entities like the age of the outgoing president of Brazil which is a real number, but not my knowledge of what this number is *Thus, not all fuzzy set membership functions are possibility distributions.* However, fuzzy intervals are possibility distributions.

Dual functions (possibility and necessity) are an inherent part of possibility theory. Dempster/Shaffer theory of plausibility and belief is another example of dual functions. The extremes of interval-valued probability are a third example of dual-type functions. Dual functions arise when there is complete information (such as probabilities on sets rather than elements or knowledge about the bounds of probability rather than the single probability distribution or knowing that we have a family of probability distribution functions that, given the information at hand equally describe our process (for example only knowledge of the support of the family of probabilities). Each of these types have a corresponding construction given next and lead to optimistic and pessimistic optimization under uncertainty. Given optimistic and pessimistic knowledge, a way to obtain a reasonable course of action is to look at minimizing the maximum regret (see [116], [117])

We are thinking of possibility theory applied to mathematical analysis, optimization in particular. To this end, we consider possibility and necessity distributions to have been constructed in one of the following ways:

1. [60] Given a set of probabilities $\Omega = \{p_\alpha(x), x \in \mathbb{R}, \alpha \in I$, where I is an index set$\}$,

$$Pos(x) = \sup_{\alpha \in I} p_\alpha(x) \tag{7}$$

$$Nec(x) = \inf_{\alpha \in I} p_\alpha(x). \tag{8}$$

2. [44] Given an unknown probability $p(x)$ which is known to exist inside a bounding pair of functions $p(x) \in [\underline{f}(x), \overline{f}(x)]$, construct necessity/possibility distributions such that

$$p(x) \in [Nec(x), Pos(x)]. \tag{9}$$

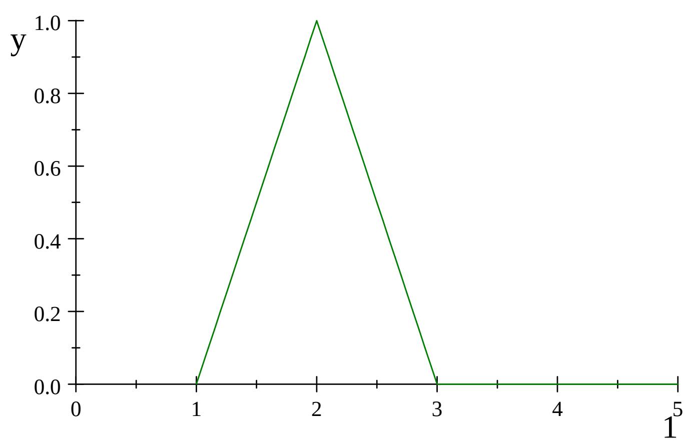
3. [8], [96] Given a probability assignment function m whose focal elements are nested, construct necessity/possibility distributions (also see [48]).

4. A fuzzy interval, defined below, generates a possibility and necessity pair. The possibility and necessity functions are constructed as was done from our initial example $[1, 4]$ depicted in Figure 1 (also see Figure 3 below).

The most prevalent approach is to define the entities of interest in optimization (the coefficients and/or the right-hand side values, for example) to be fuzzy intervals in which case they will be able to model both gradualness or transition and lack of specificity/information. Thus, possibility distributions used in possibility optimization typically are associated with the membership function of fuzzy numbers, whose generalization is called a *fuzzy interval*. If the coefficients arise from *probability-based possibility* (as in items 1-3 listed above) or dual possibility and necessity generated from a fuzzy interval, then this generates upper and lower possibility optimization (see [44]), what we have called dual distribution optimization.

Definition 5 *A* ***fuzzy number*** *is a fuzzy set with upper/lower semi-continuous membership function with one and only one value, x^*, such that $\mu(x^*) = 1$ (where x^* is the "fuzzified" number).*

For example, a fuzzy interval 2 would have uniquely $\mu(2) = 1$, which is depicted in Figure 2.A fuzzy 2 very often arises in the context of obtaining a value for a parameter based on epistemic knowledge such as in, "the value is around 2."



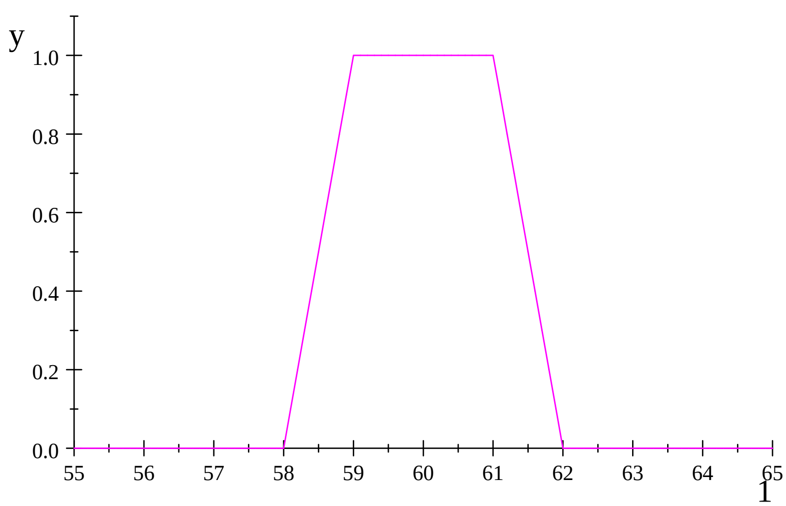
A Fuzzy number two

Definition 6 *The set of numbers for which the membership value are one is called the* ***core****.*

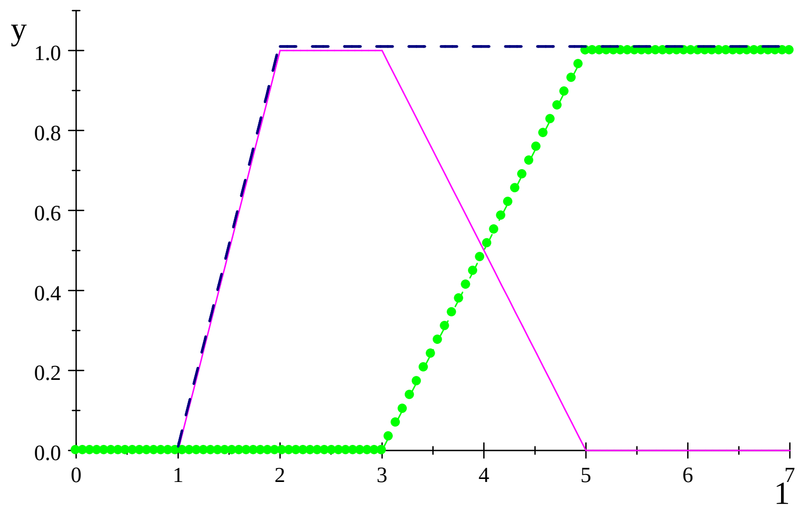
Definition 7 *A* ***fuzzy interval*** *M, depicted as a trapezoid in Figure 3, is a fuzzy number except the core (which must also exist) does not have to be a singleton but may be an interval.*

All fuzzy numbers are fuzzy intervals so we restrict our usage of fuzzy sets with numeric domains to fuzzy intervals. Possibility parameters will be fuzzy intervals.

There are various views (applications) of fuzzy intervals. A fuzzy interval can be used to enclose a set of probability distributions where the bounds are constructed from the fuzzy interval (blue line being the possibility and green line being the necessity in Figure 3). The core of the fuzzy interval is the "top," the horizontal line segment between 2 and 3 at height 1, indicated in Figure 3. The possibility and necessity as indicated below enclose all probability distribution functions whose cumulative distribution functions are bounded between the upper limit above (the possibility - dashed blue line) and lower limit below (the necessity - dotted green line). Thus, according to [9] and [12],



Fuzzy Interval - Megenta



Possibility and Necessity from a Fuzzy Interval - Megenta

"A fuzzy interval M can thus be viewed as encoding a family of probabilities, a set of probability measures P_M defined by

$$P_M = \{P|\Pi_M(A) = \sup_{a\in A} M(a) \geq prob(A), A \text{ measurable}\}.$$

It is important to notice that there are actually three probabilistic views of a fuzzy interval:
a) The *imprecise probability* view whereby M encodes a set of (cumulative) probability measures shown in Figure 3 between the dashed blue line (possibility) and the dotted green line (necessity). This is our (7) and (8).
b) The *pair of CDFs* view whereby M is defined by two random variables x^- and x^+ with cumulative distributions in blue and green of Figure 3 and M stands for the random interval $[x^-, x^+]$. This is our (9)
c) The *random set* view whereby M encodes the one point coverage function of a random interval, defined by the probability measure on the unit interval (for instance the uniformly distributed one) and a family of nested intervals (the $\alpha-$levels), via a multivalued mapping from (0,1] to $\mathbb{R}$, following Dempster [8]. This is the third approach to constructing possibility/necessity functions given above."

Let us be clear. When one has epistemically obtained (from human knowledge) a possibility description of the value of a parameter (or entity) which encapsulates what is known about the possible values of that parameter or entity, one uses a (single) fuzzy interval. If, in addition, one needs or wants bounds on all possible values based on our epistemic values of our parameters, one obtains possibility/necessity pairs via one of the four construction methods 1-4 mentioned above depending on what information can be obtained. In this latter case, for optimization, one would have to use a mini/max approach to optimization developed by [116] and [117].

A reason that one might want to use probability-based possibility (interpretations a), b) or c)) rather than probability is precisely in situations for which real values or complete probability density functions for data are not available. For example: (1) we don't know which of a given set of probabilities to use, (2) all we know is that the probability is bounded by two functions, or (3) we do not have the probability distribution on singletons, but on sets. Whether an entity of interest:

- Inherently lacks specificity (the minimal radiation that will kill a particular patient's prostate tumor cell located at (x, y, z) is $Pos(x, y, z)$), or

- Lacks sufficient research to determine its precise value even assuming that such a precise value "the minimum dosage" exists as a unique real number or its precise probability density function, or

- Its deterministic functional representation is not required, in the sense that one can get by with a more general form than its deterministic equivalent - perfect information, for the use to which it is put (the light wave reflection measured by a satellite sensor to impute the depth of the ocean, low/medium/high might suffice), or complexity reduction (low, medium, high speed for the automatic gear shifting mechanism on a car),

lack of information/specificity is a part of many if not most problems. Moreover, when we have models that are built from epistemic knowledge (human ideas about the system rather than the system itself), many of these linguistically derived models are possibility either in their entirety or partially.

Dubois and Prade [20] state:

"A set used for representing a piece of incomplete information is called a *disjunctive set.* It contrasts with a conjunctive view of a set considered as a collection of elements. A conjunctive set represents a precise piece of information. For instance, consider the quantity $v = sisters(Pierre)$ whose range is the set of subsets of possible names for Pierre's sisters. The piece of information $\{Marie, Sylvie\}$ is precise and means that Pierre's (two) sisters are Marie and Sylvie. Indeed, the frame is then $S = 2^{NAMES}$, where $NAMES$ is the set of all female first names. In this setting, a piece of incomplete information would be encoded as a disjunction of conjunctive subsets of $NAMES$."

For example, an image could be segmented/classified into two sets, stomach lining ***and*** stomach muscle. Every pixel in the image is given a value v where $0 \leq v \leq 1$ with respect to being stomach lining or stomach muscle. This is the conjunctive ***and*** thus a fuzzy set. Each pixel is stomach lining to specified degree (between 0 and 1), and (conjunction) each pixel is also stomach muscle to a specified degree (between 0 and 1). It is a mixed or conjoined set. On the other hand, suppose we use the fuzzy trapezoid interval 58/59/61/62 (see Figure 4) to model the possibility notion of a minimal tumorcidal dose to each tumor pixel. This is an incomplete set of information about each tumor pixel. There is a dosage that will kill a cancer cell (certainly that dosage that also kills the patient is a tumorcidal dosage). However, the minimal dosage that will kill cancerous cells is arguably possibility. The tumorcidal dosage is 58 ***or*** ... ***or*** 59 ***or*** ... ***or*** 61 ***or*** ... ***or*** 62. That is, a tumor pixel has an associated *distribution*, a fuzzy interval, a function representing a desired dosage at that location, the function of possible minimal dosages that will kill a tumor cell with preference at the dosages that are most likely to be lethal.

4.4 Dual Distributions: Transformations of Distributions Describing Uncertainty Entities into Dual Distributions - Possibility/Necessity

Soft constraints are assumed to have been translated into fuzzy intervals in the standard way (see for example [131]). If we have a probabilistic optimization problem whose values are known over each $x \in \mathbb{R}^n$, we would consider it under possibility optimization where the upper bound and lower bound (possibility/plausibility and necessity/belief) would be equal. If the probability were known only over sets, then we would have an upper possibility bound and a lower necessity bound as in Figure 3 and do our bound interval-valued possibility optimization which is transformed into utility optimization. A right-hand side value that is a fuzzy interval may be interpreted in two ways depending on the context of the problem. First, a fuzzy right-hand side may indicate flexibility. Second, it may indicate a possibility entity where the distribution captures the lack of specificity or available information is modeled by a fuzzy interval. For the former, the constraint becomes a flexible constraint. For the latter, it becomes a possibility constraint. For example, if we have a right-hand side value for a tumor pixel being given as the fuzzy interval 58/59/61/62 (see Figure 4) in the context of the RTP, Example 1, this fuzzy interval may represent flexibility. It may also reflect a lack of information as to the precise minimal amount radiation that will kill a cancerous cell.

4.5 Mathematical Operations on Fuzzy Intervals

We begin our section on operations of fuzzy intervals by looking at interval extension principles and its relationship to what has come to be known as Zadeh's Extension Principle both of which are an older mathematical approach.

4.5.1 Interval Extension Principle

Moore recognizes, in three Lockheed Aircraft Corporation Technical Reports, [68], [69], [70], that the extension principle is a key concept. Interval arithmetic, rounded interval arithmetic, and computing range of functions, can be derived from interval extensions. Of issue is how to compute ranges of set-valued functions. This requires continuity and compactness over interval functions, which in turn needs well-defined extension principles.

Moore in [70] uses, for the first time, an explicit extension principle for intervals called the *united extension* which particularizes set-valued extensions to sets that are intervals. Following Strother's development ([108] and [109]), a specific topological space, which includes intervals, is the starting point.

Definition 8 *A topological space* $\{X, \Omega\}$ *on which open sets are defined (*X *is a set of points, and* Ω *is family of subsets of* X*) has the property that* $X \in \Omega, \varnothing \in \Omega$*, finite intersections, and uncountable unions of sets in* Ω *are back in* Ω*. The topological space* $\{X, \Omega\}$ *is called a* $\boldsymbol{T_1}$***−space*** *if for every* $x \in X$ *and* $y \in X$ *(distinct points of* X*), there is an open set* O_y *containing* y *but not* x*. Metric spaces are* $\boldsymbol{T_1}$*. In addition, if there exists an open set* O_x *containing* x *such that* $O_x \cap O_y = \varnothing$*, then the space is called a* $\boldsymbol{T}_2$***−space****, or a Hausdorff space. The set of subsets of* X *is denoted by* $S(X)$*.*

Lemma 1 *[109] If* $f : X \to Y$ *is a continuous multi-valued function and* Y *is a* $\boldsymbol{T_1}$***−space****, then* f *is closed for all* x*. That is, the image of a closed set in* X *is closed in* Y*.*

Remark 5 *This means that* f *is a point-closed function. In other words, when the sets of* X*,* $S(X)$*, are retracted to points, then the set-valued function* F *on* $S(X)$ *becomes the original function* f *when defined through the retract. Another way of putting this is, if* f *is a real-valued function, then the mapping extended to sets of* $X \subseteq \mathbb{R}$*,* $S(X)$*, where the range is endowed with a* $\boldsymbol{T_1}$ *topology, is well-defined.*

Remark 6 *[70] If* $f : X \to Y$ *is an arbitrary mapping from an arbitrary set* X *into an arbitrary set* Y*, the united extension of* f *to* $S(X)$*, denoted* F*, is defined as follows.*

$$\begin{aligned} F &: S(X) \to S(Y) \text{ where} \\ F(A) &= \{f(a) \mid \forall\, a \in A, A \in S(X)\}, \quad \text{in particular} \\ F(\{x\}) &= \{f(x) \mid x \in \{x\}, \{x\} \in S(X)\} \text{ where } \{x\} \text{ is a singleton set.} \end{aligned}$$

Thus,

$$F(A) = \bigcup_{a \in A} f(a).$$

This definition, as we shall see, is quite similar to the fuzzy extension principle of Zadeh, where the union is replaced by the supremum.

Theorem 2 *[109] Let* X *and* Y *be compact Hausdorff spaces and* $f : X \to Y$ *continuous. Then the united extension of* f*,* F*, is continuous. Moreover,* F *is closed.*

There are fixed "point" (set) theorems associated with these set-valued maps, the united extensions, which can be found in [109] and not repeated here since we are not interested in the full generality of these theorems, but only as they are related to intervals. Moore ([70], [71], [73]) particularized the ideas of Strother ([108], [109]) to spaces consisting of subsets of $\mathbb{R}$ that are closed and bounded (real intervals), which we will denote $S([\mathbb{R}])$, where $S([X])$ denotes the set of all intervals on any set of real numbers $X \subseteq \mathbb{R}$. To this end, Moore had to develop a topology on $S([\mathbb{R}])$ that proved to be Hausdorff. The results of interest associated with the united extension for intervals are the following [73].

1. **Isotone Property**: A mapping f from a partially ordered set (X, r_X) into another (Y, r_Y) where r_X and r_Y are relations, is called *isotone* if $x\ r_X\ y$ implies $f(x)\ r_Y\ f(y)$. In particular, the united extension is isotone with respect to intervals and the relation $\subseteq$. That is, for $A, B \in S([X])$, if $A \subseteq B$, then $F(A) \subseteq F(B)$.

2. **The Knaster-Tarski Theorem**: An isotone mapping of a complete lattice into itself has at least one fixed point.

The Knaster-Tarski theorem implies that F, the united extension, $F : S([\mathbb{R}]) \to S([\mathbb{R}])$, has at least one fixed "point" (set) in $S([\mathbb{R}])$, which may be the empty set, and has an important numerical consequence. Consider the sequence $\{X_n\}$ in $S(X)$ defined by

$$X_0 = X$$

$$X_{n+1} = F(X_n).$$

Since

$$X_1 \subseteq F(X_0) = F(X) \subseteq X = X_0,$$

then, by induction,

$$X_{n+1} \subseteq X_n.$$

Let

$$Y = \bigcap_{n=0}^{\infty} X_n.$$

The following is true [70]. If $x = f(x)$ is any fixed point of f in X, then $x \in X_n$ for all $n = 0, 1, 2, \ldots$ so that $x \in Y$ and

$$x \in F(Y) \subseteq Y.$$

Thus, X_n, Y, and $F(Y)$ contain all the fixed points f in X. If Y and/or $F(Y)$ is empty, then there are no fixed points of f in X. Newton's method is a fixed point method, so that the above theorem pertains to a large class of problems. Moreover, these enclosures lead to computationally verified solutions when implemented on a computer with rounded interval arithmetic.

4.5.2 Interval Arithmetic

Interval arithmetic was defined by R. C. Young [126] in 1931, P. S. Dwyer [22] in 1951, M. Warmus [122] in 1956, and then independently by T. Sunaga [111] in 1958. Moore ([68], [69]) rediscovers and extends interval arithmetic to rounded interval arithmetic, thereby allowing interval arithmetic to be useful in computational mathematics. In addition, Moore developed interval analysis ([73]). There are two approaches to interval arithmetic. The first is the interval arithmetic obtained by application of the united extension. The second approach is the standard approach. There is an interval arithmetic and associated semantics that allows for "intervals," $[a, b]$, for which $a > b$ ([28], [34]). This arithmetic is related to directed interval arithmetic and has some interesting applications to fuzzy control ([3], [95]). This generalization is not pursued further. Nevertheless, it has an intimate relationship to solving interval (fuzzy) linear system of equations and inequalities.

The basic definitions associated with interval arithmetic are fully developed in [73]. As is well known, the definitions for interval arithmetic for intervals are:

1. $[a, b] + [c, d] = [a + c, b + d]$
2. $[a, b] - [c, d] = [a - d, b - c]$
3. $[a, b] \times [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$
4. For $0 \notin [c, d], [a, b] \div [c, d] = [\min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}]$

Definition 9 *The interval arithmetic associated with 1-4 above we call* ***standard interval arithmetic (SIA)****.*

There are various properties associated with SIA which are different from those of real numbers and that of the constraint interval arithmetic (CIA) defined subsequently. In particular, interval arithmetic derived from the definitions above, SIA, is subdistributive. Thus, from [74] we have for intervals $[x], [y]$, and $[z]$:

1. $[x] + ([y] + [z]) = ([x] + [y]) + [z]$ - the associative law for addition
2. $[x] \cdot ([y] \cdot [z]) = ([x] \cdot [y]) \cdot [z]$ - the associative law for multiplication
3. $[x] + [y] = [y] + [x]$ - the commutative law for addition
4. $[x] \cdot [y] = [y] \cdot [x]$ - the commutative law for multiplication
5. $[0,0] + [x] = [x] + [0,0] = [x]$ - additive identity
6. $[1,1] \cdot [x] = [x] \cdot [1,1] = [x]$ - multiplicative identity
7. $[x] \cdot ([y] + [z]) \subseteq [x] \cdot [y] + [x] \cdot [z]$ - the subdistributive property

Example 3 *[74], page 13, points out that*

$$[1,2](1-1) = [1,2](0) = 0,$$

whereas

$$[1,2](1) + [1,2](-1) = [-1,1].$$

Remark 7 *From Moore's [68] implementation of [111] (neither Moore nor Sunaga seem to have been aware of Warmus' earlier work [122]), he states that*

$$[x] \circ [y] = \{z \mid z = x \circ y, x \in X, y \in [y], \circ \in \{+, -, \times, \div\}\}$$

which means that Moore applies the united extension for distinct intervals $[x]$ *and* $[y]$. *However, Moore abandons this united extension definition and only uses SIA. The definitions for SIA lead to a simplification of the operations since one does not have to account for multiple occurrences, while at the same time it leads to overestimation which is severe at times. From the beginning, Moore was aware of the problems of overestimation associated with multiple occurrences of the same variable in an expression. Moreover, it is apparent that, from the standard approach,* $[x] - [x]$ *is* ***never*** 0 *unless* $[x]$ *is a real number (a zero width interval). Moreover,* $[x] \div [x]$ *is* ***never*** 1 *unless* $[x]$ *is a real number (a zero width interval).*

4.5.3 Standard Interval Arithmetic, SIA

The standard approach to interval arithmetic considers all instantiations of variables as independent. That is, the Young [126], Warmus [122], Sunaga [111], and Moore [71] standard approach to interval arithmetic is one in which multiple occurrences of a variable in an expression are considered as independent variables. While SIA is quite simple to implement, it leads to overestimations.

Example 4 *Consider*

$$f(x) = x(x-1),\ x \in [0,1].$$

Using the axiomatic approach,

$$[0,1]([0,1]-1) = [0,1][-1,0] = [-1,0]. \tag{10}$$

However, the interval containing $f(x) = x(x-1)$ *is* $[-0.25, 0]$. *This is because the two instantiations of the variable* x *are taken as independent when they are dependent. The united extension* $F(A)$, *which is*

$$F([0,1]) = \bigcup_{x \in [0,1]} \{f(x)\} = [-0.25, 0],$$

was not used.

If the calculation were $x(y-1)$ for $x \in [0,1]$, $y \in [0,1]$, then the tightest interval containing $x(y-1)$, its united extension, indeed is $[-1,0]$. Note that the subdistributivity property does not use the united extension in computing $[x] \cdot [y] + [x] \cdot [z]$ but instead considers $[x] \cdot [y] + [w] \cdot [z]$ where $[w] = [x]$. Partitioning the interval variables (that are repeated) leads to tighter approximations to the united extension. That is, take the example above and partition the interval in which x lies.

Example 5 *Consider* $x(x-1)$ *again, but* $x \in [0,0.5] \cup [0.5,1]$. *This yields*

$$[0,0.5]([0,0.5]-1) \cup [0.5,1]([0.5,1]-1) = \tag{11}$$

$$[0,0.5][-1,-0.5] \cup [0.5,1][-0.5,0] = \tag{12}$$

$$[-0.5,0] \cup [-0.5,0] = [-0.5,1] \tag{13}$$

which has an overestimation of 0.25 *compared with an overestimation of* 0.5 *when the full interval* $[0,1]$ *was used.*

In fact, for operations that are continuous functions, a reduction in width leads to estimations that are closer to the united extension and in the limit, to the exact united extension value ([71], [73], [74], [80]). There are other approaches which find ways to reduce the overestimation arising from the standard approach that have proved to be extremely useful such as centered, mean value, and slope forms ([33], [46], [74], [80], [82], [88]).

4.5.4 Interval Arithmetic from the United Extension: Constraint Interval Arithmetic

The power of SIA is that it is simple to apply. Its complexity is at most four times that of real-valued arithmetic. However, SIA leads to overestimations in general because it takes every instantiation of the same variable independently. The united extension, when applied to sets of real numbers is global optimization, which in the general case, is *NP-Hard*. However, the same thing is true for SIA when applied to function evaluations in which partitioning or a partitioning-based method is required to obtain realistic and tightest bounds. Moreover, simple notions such as

$$[x] - [x] = 0 \text{ and} \tag{14}$$

$$[x] \div [x] = 1, 0 \notin [x] \tag{15}$$

are desirable properties and can be maintained if the united extension is used to define interval arithmetic (see [50]). In the context of fuzzy arithmetic which uses interval arithmetic, Klir [47] looked at fuzzy arithmetic which was constrained to account for (14) and (15) from a case-based approach. What is given next was developed in [50] independently of [47] and is more general than the case-based method. Constraint interval arithmetic is derived directly from the united extension rather than standard or case-based.

It is known that applying interval arithmetic to the union of intervals of decreasing width yields tighter bounds on the result that converges to the united extension interval result [71]. Of course, for n-dimensional problems, "intervals" are rectangular parallelepipeds (boxes), and as the diameters of these boxes approach zero, the union of the result approaches the correct bound for the expression. Partitioning each of the sides of the n-dimensional box in half has complexity of $O(2^n)$ for each split resulting in an exponential complexity. Theorems proving convergence to the exact bound of the expression and the rates associated with the subdivision of intervals can be found in [33], [46], [74], [80], [82] or [88]. What is proposed here is to redefine interval numbers in such a way that dependencies are explicitly kept. The ensuing arithmetic will be called *constraint interval arithmetic* (CIA). This new arithmetic is the derivation of arithmetic directly from the united extension of [108].

An interval number is redefined [50] next into an equivalent form next as the real-valued function of one variable and two coefficients or parameters.

Definition 10 *An interval* $[\underline{x}, \overline{x}]$ *is the real single-valued function* $X^I(\lambda_x)$*, where*

$$\begin{aligned} X^I(\lambda_x) &= (1-\lambda_x)\underline{x} + \lambda_x \bar{x}, \\ &= w_x \lambda_x + \underline{x},\ 0 \le \lambda_x \le 1, \end{aligned} \tag{16}$$

where $w_x = \bar{x} - \underline{x} \geq 0$ is the width of the interval. Strictly speaking, in (16), since the numbers $\underline{x}$ and $\bar{x}$ (consequently w_x) are known (inputs or data), they are **coefficients**, whereas λ_x is varying, although constrained between 0 and 1, hence the name "constraint interval arithmetic." This means that $X^I(\lambda_x)$ is a single-valued real linear function with two coefficients.

Remark 8 *Intervals in the sense of CIA are linear functions with non-negative slopes over the domain* $[0,1]$*. Thus, the space to which the CIA representation of intervals belongs is the space of single-valued linear functions with non-negative slopes over a compact domain. Thus, CIA algebra is* ***not*** *the same as that of SIA. That is, (16) CIA arithmetic is single-valued function arithmetic over the compact domain* $[0,1]$*. SIA is an arithmetic of a two component vector* $[x] = (x_1, x_2), x_1 = \underline{x} \leq x_2 = \bar{x}$*, over a subset of* $\mathbb{R}^2$*, the upper half-plane determined by the line* $x_2 = x_1$*. Of course, the additive inverses of SIA do not belong to the space (upper half-plane determined by the line* $x_2 = x_1$*) unless the components are equal, the interval is a zero-width interval, a real number. Additive inverses belong to the lower half-plane in* $\mathbb{R}^2$ *determined by the forty-five degree line* $x_1 = x_2$*.*

The algebraic operations for CIA are defined as follows.

$$
\begin{aligned}
[z] &= [\underline{z}, \bar{z}] = [x] \circ [y] && (17)\\
&= \{z \mid z = x \circ y,\ \forall\ x \in X^I(\lambda_x),\ y \in Y(\lambda_y), 0 \le \lambda_x, \lambda_y \le 1\}\\
&= \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) \circ ((1-\lambda_y)\underline{y} + \lambda_y \bar{y})), 0 \le \lambda_x \le 1, 0 \le \lambda_y \le 1\}\\
\text{where } \underline{z} &= \min\ \{z\},\ \bar{z} = \max\ \{z\},\\
0 &\le \lambda_x \le 1, 0 \le \lambda_y \le 1 \text{ and } \circ \in \{+,-,\times,\div\}. && (18)
\end{aligned}
$$

Remark 9 *It is clear from (18) that constraint interval arithmetic is a constrained global optimization problem. However, when the operations use the same interval, no exceptions need to be made as in [47] nor in more intelligent applications of SIA.*

Thus, we the operation for the double occurrence of the same variable is obtained by using only (17),

$$
\begin{aligned}
[z] &= [\underline{z}, \bar{z}] = [x] \circ [x] && (19)\\
&= \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) \circ ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\}.
\end{aligned}
$$

This results in the following properties.

1. Addition of the same interval variable:

$$
\begin{aligned}
[x]+[x] &= \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) + ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\}\\
&= \{z \mid z = 2((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\} = [2\underline{x}, 2\bar{x}].
\end{aligned}
$$

2. Subtraction of the same interval variable:

$$
[x]-[x] = \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) - ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\} = 0.
$$

3. Division of the same interval variable, $0 \notin [x]$:

$$
[x]\div[x] = \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) \div ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\} = 1
$$

4. Multiplication of the same interval variable with $\underline{x} < \bar{x}$

$$
\begin{aligned}
[x]\times[x] &= \{z \mid z = ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}) \times ((1-\lambda_x)\underline{x} + \lambda_x \bar{x}), 0 \le \lambda_x \le 1\}\\
&= \{z \mid z = ((1-\lambda_x)^2 \underline{x}^2 + 2(1-\lambda_x)\lambda_x \underline{x}\bar{x} + {\lambda_x}^2 \bar{x}^2, 0 \le \lambda_x \le 1\}\\
&= [\min\{\underline{x}^2, \bar{x}^2, 0\}, \max\{\underline{x}^2, \bar{x}^2, 0\}].
\end{aligned}
$$

To verify that this is the interval solution, note that as a function of the single variable λ_x, the product, $[x] \times [x]$, is

$$f(\lambda_x) = (\bar{x} - \underline{x})^2 \lambda_x^2 + 2\underline{x}(\bar{x} - \underline{x})\lambda_x + \underline{x}^2,$$

which has a critical point at

$$\lambda_x = -\frac{\underline{x}}{\bar{x} - \underline{x}} = -\frac{\underline{x}}{w_x}.$$

Thus

$$\begin{aligned} \underline{z} &= \min\{f(0), f(1), f(-\frac{\underline{x}}{w_x})\}, \bar{z} = \max\{f(0), f(1), f(-\frac{\underline{x}}{w_x})\} \\ &= \min\{\underline{x}^2, \bar{x}^2, 0\}, \bar{z} = \max\{\underline{x}^2, \bar{x}^2, 0\} \end{aligned}$$

as is obvious. Of course, if $\underline{x} = \bar{x}$, then $[x] \times [x] = x^2$.

5. $[x]([y] + [z]) = [x][y] + [x][z]$.

CIA is the complete implementation of the united extension, and possesses an algebra which has an additive inverse, a multiplicative inverse, and a distributive law. Of course, this is because the representation of intervals in the CIA sense live in a special space of single-valued linear functions with non-negative slopes over the compact domain $[0, 1]$.

Affine Arithmetic Another more recent approach to minimize the effects of overestimation due to dependencies is affine arithmetic [106]. A number x, whose value is subject to uncertainty, has the representation

$$x = x_0 + x_1\epsilon_1 + ... + x_n\epsilon_n, \tag{20}$$

where x_i are coefficients (known, real) and the $\epsilon_i \in [-1, 1]$. That is, x_i represents the magnitude of "error" and ϵ_i represents the ith uncertainty that is contributing to the total uncertainty represented by the interval. To recover an interval from the affine representation of a number (20), each of the ϵ_i is replaced by $[-1, 1]$ so that

$$X = [x_0 - \xi, x_0 + \xi], \tag{21}$$

where $\xi = \sum_{i=1}^{n} |x_i|$. Clearly, if one is given an interval $X = [\underline{x}, \overline{x}]$, to obtain the affine number representation (20),

$$x = x_0 + x_1 \epsilon_i,$$

where $x_0 = (\underline{x} + \overline{x})/2, x_r = (\underline{x} - \overline{x})/2$, and $\epsilon_i \in [-1, 1]$, where the subscript is used to distinguish the variable x_0 from all other variables.

Example 6 *Suppose $x = 4 + 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$, and $y = 2 + \epsilon_1 - 3\epsilon_2 + \epsilon_4$. From this representation, we see that x depends on variables 1, 2, and 3 whereas y depends on variables 1, 2, and 4. Note that the dependency information is carried forward, where $X = [-2, 10]$, and $Y = [-3, 7]$. Interval analysis which does not carry forward dependencies would obtain a sum of*

$$Z = X + Y = [-2, 10] + [-3, 7] = [-5, 17].$$

Affine arithmetic yields

$$z = x + y = 6 + 4\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4,$$

from which we obtain $Z = [-1, 13]$.

Addition and subtraction of affine numbers is straight forward. However, the challenge comes with multiplication and division in such a way that the interval result is guaranteed to enclose the true result. Moreover, obtaining an affine number for arbitrary transcendental functions, and composites of these, is another challenge. These have been worked out for interval arithmetic in such a way that roundoff errors are incorporated in the final result. The challenge in affine arithmetic as it is in quantile, range, and generalized interval arithmetic is how to implement multiplication and division. These matters can be found in [106].

Multiplication [106]

$$\begin{aligned} z &= xy \\ &= (x_0 + \sum_{i=1}^{n} x_i\epsilon_i)(y_0 + \sum_{i=1}^{n} y_i\epsilon_i) \\ &= x_0y_0 + \sum_{i=1}^{n}(x_0y_i + y_0x_i)\epsilon_i + (\sum_{i=1}^{n} x_i\epsilon_i)(\sum_{i=1}^{n} y_i\epsilon_i) \qquad (22) \\ &= z_0 + \sum_{i=1}^{n} z_i\epsilon_i + Q(\epsilon_1, ..., \epsilon_n), \qquad (23) \end{aligned}$$

where $z_i = x_0 y_i + y_0 x_i$, and

$$\begin{aligned} Q(\epsilon_1, ..., \epsilon_n) &= (\sum_{i=1}^{n} x_i \epsilon_i)(\sum_{i=1}^{n} y_i \epsilon_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \epsilon_i \epsilon_j && (24) \\ &\subseteq \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j [-1, 1][-1, 1] && (25) \\ &= [-1, 1] \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j && (26) \\ &= [\underline{q}, \overline{q}]. && (27) \end{aligned}$$

To obtain an affine representation of (22) that contains the product,

$$\begin{aligned} z &= \hat{z}_0 + \sum_{i=1}^{n} z_i \epsilon_i + (\underline{q} + \overline{q})/2 + ((\underline{q} - \overline{q})/2)\epsilon_{n+1} \\ &= z_0 + \sum_{i=1}^{n+1} z_i \epsilon_i, \end{aligned}$$

where $z_0 = \hat{z}_0 + (\underline{q} + \overline{q})/2$, and $z_{n+1} = (\underline{q} - \overline{q})/2$. The authors of [106] go on to successfully and efficiently apply affine arithmetic to computer graphics.

5 An Approach to Interval, Fuzzy Set Theory, and Possibility Theory in Optimization

An appropriate classification of *interval optimization* and *possibility optimization* is *optimization under uncertainty,* where some (or all) the input data (parameters) to the optimization model lack specificity, and/or the information is insufficient to yield a real valued number or a probability distribution. This is distinguished from *fuzzy optimization* that appropriately belongs in the class of *flexible programming* problems. The uncertain parameters that are possibility means that more information about the parameter can never yield less certainty since possibility has the axiomatic property that if $A \subseteq B, P(A) \leq P(B)$. This property (more information never leading to greater uncertainty) is the isotone property and not *a-priori* present in fuzzy measures.

There are two broad approaches to dealing with uncertainty in the parameters associated with the constraint set. We are assuming that standard methods for converting soft constraints into fuzzy intervals have been used. The first is to consider the constraint set as possibility and move them into the objective function where violations are penalized much as is done in defining a Lagragian and optimizing the Lagragian ([43], [62]). The second is to compute the constraints at each α-level and treating constraints similar to chance constraints ([37]). The first approach is the focus of the next section, "Interval Optimization." The second approach is a generalization of interval optimization approach is found in the section, "Fuzzy and Possibility Optimization and Semantics."

5.1 Interval Optimization

Intervals can represent uncertainty in a parameter value (a, b, c, d, e of (1),(2) and (3)). This is the point of view of ([36], [38], [59], [60], [97]). However, [49], [66], [67], [82], [89], [90], [104], [105]) approach interval linear programs from a strictly interval perspective, where [24] use both views. Historically, however, intervals have been used in global optimization (see [33], [46], [88]). In fact, in a recent email communication with Arnold Neumaier [83], he says:

> "From my point of view, interval analysis is part of deterministic global optimization. Its aim is finding good, provable bounds on ranges [of functions], which is finding optimal, probable bounds on ranges. One needs global optimization techniques to get good bounds, and one needs interval techniques to get good global optimization certificates. Thus, they are happily married, and it is this view of intervals that makes them respectable in the numerical analysis community."

Our point of view for this exposition is that intervals may be used in and of themselves (in obtaining provable bounds for global optimization problems) and to capture uncertainties where all that is known is the bounds on parameters. However, we will not develop further the former view except as it is used in obtaining bounds on constraints. When the uncertainty in the rim parameter c (the set of parameters associated with the objective function) is interval, we will transform the problem into a possibility optimization problem. So, we put this discussion off for the "Possibility Optimization" section.

5.2 Fuzzy and Possibility Optimization and Semantics

Next what is meant by decision-making in the presence of fuzzy and possibility entities is defined. These definitions are central to the semantics and methods. In their book (Chapter 5) Dubois and Prade [19] give clear definitions and distinctions of fuzzy measures, possibility and probabilities often forgotten and ignored by researchers (see also Chapters 1 and 7 of [19] and more recently [11]). We use these distinctions in formulating and obtaining solution methods in the context of optimization.

1. *Fuzzy Decision Making:* Given the set of real valued (crisp) decisions, Ω, and fuzzy sets, $\{\tilde{F}_i \mid i = 1 \text{ to } n\}$, find the optimal decision in the set Ω. That is,
$$\sup_{x\in\Omega} h\left(\tilde{F}_1(x), ..., \tilde{F}_n(x)\right), \tag{28}$$
where $h : [0,1]^n \to [0,1]$ is an aggregation operator [48], often taken to be the *min* operator, and $\tilde{F}_i(x) \in [0,1]$ is the fuzzy membership of x in fuzzy set $\tilde{F}_i$. The decision space Ω is a set of real numbers (*crisp set*), and the optimal decision satisfies a mutual membership condition defined by the aggregation operator h. This is the method of Bellman and Zadeh [1], Tanaka, Okuda and Asai [113], [114], and Zimmermann [131], who were the first (in this order) to develop fuzzy mathematical programming. While the aggregation operator h historically has been the *min* operator, it can be, for example, any $t{-}norm$ that is consistent with the context of the problem and/or decision methods (see [45] or [93]).

2. *Possibility Decision Making:* Given the set of real valued (crisp) decisions, Ω, and the set of possibility distributions representing the uncertain outcomes from selecting decision $\vec{x} = (x_1, ..., x_n)^T$ denoted $\Psi_x = \{\hat{F}^i_x, i = 1, \ldots, n\}$, find the optimal decision that produces the best set of possible outcomes with respect to an ordering U of the outcomes. That is,
$$\sup_{\Psi_x\in\Psi} U(\Psi_x), \tag{29}$$
where $U(\Psi_x)$ represents an "evaluation **function**" (utility) of the set of distributions of possible outcomes $\Psi = \{\Psi_x | x \in \Omega\}$. The decision space Ψ is a **set of possibility distributions** $\Psi_x : \Omega \to [0,1]$ resulting from taking decision $x \in \Omega$. This is the semantic taken in the possibility

optimization of Inuiguchi [35], [37], [39] and Jamison and Lodwick [43]. If $\hat{F}_x = \hat{2}x_1 + \hat{3}x_2$, where $\hat{2}$ and $\hat{3}$ are the possibility numbers 2 and 3, then each $\vec{x} = (x_1, x_2)^T$ generates the possibility distribution $\hat{F}_x = \hat{2}x_1 + \hat{3}x_2$.

Remark 10 *Let us summarize what we have just stated because we consider this an important point often forgotten. For fuzzy sets $\tilde{F}_i$, $i = 1, ..., n$, given x, $[\tilde{F}_1(x), ..., \tilde{F}_n(x)]^T$ is a real valued* ***vector****. Thus, we need a way to aggregate the components of the vectors into a single real value. This is done by a t-norm, min for example. For possibility, given x, $\Psi_x = \{\hat{F}_x^i, i = 1, \ldots, n\}$ is a set of* ***distributions****, so we need a way to turn this set of distributions into a single real value. This is may be implemented using an evaluation function, a generalized expectation with recourse, or stochastic dominance, for example.*

Very simply, fuzzy decision-making selects from a set of real valued, crisp, elements ordered by an aggregation operator on corresponding membership functions, while possibility decision making selects from a set of distributions measured by a utility operator that orders the corresponding distributions. These two different approaches have two different ordering operators (an aggregation operation for fuzzy sets such as *min* and a utility function in the case of possibility such as a generalized expectation) and lead to two different optimization methods (see [58]). The underlying sets associated with fuzzy decision-making are fuzzy, where one forms the decision space of real valued elements from operations ("*min*" and "*and*", for example, in the case of optimization of [1], [114] and [131]) on these fuzzy sets. The underlying sets associated with possibility decision making are real valued sets, where one forms the decision space of (possibility) *distributions* from operations on real valued sets.

The construction of an appropriate evaluation function is a challenge. The axioms of utility theory as developed by Von Neumann and Morgenstern [120] are usually required. The type of utility function that is used is a challenge and decision maker dependent. For example, if one is radiating a tumor that is quite aggressive, one's utility might have higher risk (the first derivative is large and positive over the domain) than if one were radiating a tumor that was growing very slowly (the first derivative is small and positive over the domain). For this presentation, we put aside the question of how to obtain an appropriate utility function noting that it is a key to the successful implementation of the methods. The key point is that in possibility optimization, one is using a utility such as a generalization of the expectation with recourse to transform distributions into one real valued function (which is then optimized), whereas in fuzzy optimization, one is using an aggregation operator such as a *min* or *t-norm* to transform vectors into one real valued function (which is then used in the optimization).

The idea of the use of utility for decision making under uncertainty problems is discussed in [21] who show how to use two qualitative counterparts to the expected utility criterion, one type of utility, U, that can be used in (29), to express uncertainty and preferences in decision making under uncertainty. Thus, what is called here *possibility decision making,* (29), is related to what [21] developed. However, optimization as articulated here are quantitative methods (the mapping $U : \Psi \to \mathbb{R}$), whereas the focus of [21] is more qualitative.

5.3 Fuzzy Decision Making Using Fuzzy Optimization - Flexible Optimization

Fuzzy decision making using fuzzy optimization was first operationalized by Tanaka, Okuda, and Asai (see [113], [114]) and then by Zimmermann (see [131]). This approach, based on the landmark theoretical paper by Bellman and Zadeh [1], relaxes systems of inequalities $Ax \leq b$ and the objective function to denote aspirations. The results are *soft constraints,* where the number b to the right of the soft inequality is a target such that, if the constraint is less than or equal to b, the membership value is one (the constraint is satisfied with certainty), and, if the constraint is greater than $b + d$, (for an *a-priori* given $d \geq 0$), the membership is zero (the constraint is not satisfied with certainty). In between, the membership function is interpolated so that it is consistent with the definition of a fuzzy interval membership function in the context of the problem. Linear interpolation was the original form (see [131]). This models a fuzzy meaning of inequality that is translated into a fuzzy membership function and is the source of our use of the designation of **flexible programming** for these classes of problems. The $\alpha - level$ represents the degree of feasibility of the constraints, consistent with the aspiration that the inequality be less than b but definitely not more than $b + d$. Thus, the objective (according to [131]) is to simultaneously satisfy all constraints at the highest possible level of feasibility as measured by the $\alpha - levels$ of the membership functions (that is "and" all membership functions).

The approach of Tanaka, Okuda, and Asai (see [113], [114]) and Zimmermann (see [131]) deals with one way to minimize constraint violations. However, their operationalization is not always Pareto optimal [10]. Their approach must be iterated - fix the constraints at bounds and re-optimize. There is another related form of flexibility and that is allowing constraint violations to be measured by a possibility (necessity) or probability distribution [37].

These methods falls within a goal satisfaction approach in optimization in which the highest degree of goal attainment is sought. They do this by minimizing the violation of the most stringent constraint. Thus, for example, this approach may guarantee that every constraint is satisfied to a 0.65 degree or more, and it may be the case that every constraint is satisfied to the 0.65 level. However, it may also be that if one of the constraints were relaxed to a 0.6 constraint violation level, all others may be satisfied at a 0.95 level. That is, this approach does not look at the aggregate satisfaction, only the most constraining one. It is minimizing the maximum constraint violation.

An aggregate goal attainment tries to maximize an overall *measure* of aggregate goal satisfaction. The *aggregate* sum of goal attainment focuses on maximizing the cumulative satisfaction of the goals. The surprise function (see [62], [81]) is one such measure for an aggregate set of (fuzzy) goals. In particular, when the right-hand side values are interpreted as goals rather than rigid constraints, the problem may be translated into one of optimizing the aggregate goal satisfaction. Thus, for soft constraints derived as,

$$\text{hard } \; y_i = (A\vec{x})_i \leq b_i \Rightarrow \text{soft } \; y_i = (A\vec{x})_i \leq \tilde{b}_i, \tag{30}$$

where the right-hand side values of the soft constraint are *fuzzy intervals,* the transformation into a set of aggregate goal satisfaction problem using the surprise function as the measure for the cumulative goal satisfaction is attained as follows. A (soft) fuzzy inequality (30) is translated into a fuzzy membership function, $\mu_i(x)$, which is the possibility $pos(\tilde{b}_i \geq Ax)$. Each membership function is translated into a surprise by

$$s_i(x) = (\frac{1}{\mu_i(x)} - 1)^2. \tag{31}$$

These functions are added to obtain a total surprise

$$S(\vec{x}) = \sum_i s_i((A\vec{x})_i). \tag{32}$$

Note that (32) is an aggregation operator. A best compromise solution based on the surprise function is given by the nonlinear optimization problem

$$\begin{aligned} \min\ z &= S(\vec{x}) = \sum_i s_i((A\vec{x})_i) \\ \text{subject to } x &\in \Omega \text{ (possible hard constraints).} \end{aligned}$$

That is, a real valued inequality constraint whose right-hand side value is a fuzzy interval is translated into a fuzzy set. This fuzzy set is then transformed into a real-valued function, $S(\vec{x})$. The objective is to minimize the **sum** of all surprise function values, $S(\vec{x})$. Unlike Tanaka and Zimmermann, the constraints are not restricted such that *all* satisfy a minimal level. The surprise function approach effectively sums each of the $\alpha - levels$ for each of the constraints, then maximizes this sum with respect to α. Since the optimization is over sets of crisp values coming from fuzzy sets, the surprise approach is a fuzzy optimization method. The salient feature is that surprise uses a dynamic penalty for falling outside distribution/membership values of one. The advantage is that the individual penalties are convex functions, which become infinite as the values approach the endpoints of the support. Moreover, this approach is computationally tractable [62].

Remark 11 *The surprise approach may be used to handle soft constraints of Tanaka, Okuda, and Asai (see [114]) and Zimmermann (see [131]), since these soft constraints can be considered to be fuzzy intervals. However, if soft constraints are handled using surprise functions, the sum of the failure to meet the constraints is minimized rather than forcing each constraint to meet a minimal (fuzzy) feasibility level or a predetermined feasibility level. There are other approaches. A more recent set of papers can be found in [61].*

5.4 Single Possibility Distribution Decision Making

One approach to possibility distributions of parameters ([43] and [54]) allows all constraint violations at an established cost or penalty and minimizes the expected average, a generalization of expected value ([15], [41], [43], [124], and [125]). This approach considers all possible outcomes as a weighted expected average penalty. The expected average may be considered to be a type of utility. This particular utility takes violations as penalties on all outcomes of the constraints. It optimizes over sets of possibility distributions, so it is possibility optimization.

Another approach that optimizes over possibility distributions [37] and [39]also optimize over distributions considers constraint feasibility as possibility generalizations of *chance constraint* methods. The approach used in [43] and [54] is a possibility generalization of the *recourse models* in stochastic optimization (see for example [2]), where violations of constraints are considered as allowable up to a maximum but at a cost. The recourse model in the context of non-probabilistic uncertainty has been studied by [40] where interval parameters/coefficients are treated.

5.5 Dual Possibility Distribution Decision Making Under Uncertainty

Dual distribution evaluation optimization is an optimization in which upper and lower bounds on uncertainty are used. The upper bounds yield optimistic results while the lower bounds yield pessimistic results. Methods that yield minimum maximum regret have been developed by [116] and [117].

5.6 Mixed Fuzzy and Possibility Decision Making: Mixed Possibility and Probability Optimization Methods

An optimization problem containing a mixture of uncertainty together in one or more constraints is called *mixed optimization* according to our taxonomy. Problems in which one type of uncertainty or flexibility occurs in a parameter of a constraints and another uncertainty or flexibility parameter occurs in another constraint (each constraint having a single type of uncertainty or flexibility) has been studied (see [39], [59]). There are other problems in which two or more types of uncertainty occur in two or more parameters of a single constraint. Thus, within a quantitative setting, there are two cases for the mixed problem. The first case is a problem that contains several uncertainty (or fuzzy/flexible) parameters (or soft inequalities) or uncertainty (possibility, necessity, interval, upper/lower probability, belief/plausibility), but each type is solely in one constraint.. In this case, the fuzzy constraints can be optimized by $\alpha - levels$ (according to [81] or [131]) and the possibility constraints penalized according to [43]. The second case in which two or more uncertainty parameters appear in the same constraint, one must consider a generalized theory such as interval probabilities or random sets (see [59], [60], [116], or [117]). An effect way to solve mixed problems, once it is translated into a generalized over arching uncertainty theoretical setting such as random sets, is to state it as a generalized recourse model. This is the approach found in [116] and [117].

6 Constraint Set Under Uncertainty

There are two broad types of constraint sets in the presence of fuzziness (flexibility) and uncertainty (interval, probability, possibility). We focus on interval uncertainty since these, if combined with constraint interval arithmetic or the Extension Principle of Zadeh will lead directly into fuzzy linear system analysis.

6.0.1 Linear Interval Equations

We denote an $m \times n$ interval matrix by $[A]$ which is a matrix with interval entries and an $m \times 1$ interval vector by $[b]$ which is a vector with interval entries. In light of what we mentioned in terms of equality of interval equations not being equivalent to pairs of inequalities, we define two different linear problems.

Definition 11 *Let*

$$[A]x = [b] \tag{33}$$

$$[A]x \leq [b] \tag{34}$$

(33) and (34) are ***systems of linear interval equations and inequalities*** *respectively.*

Parameters, $[A]$ and $[b]$, and solutions, $x \in \Omega$, of the linear algebraic system of equations under interval uncertainty have a dual nature in that they possess **numeric** properties (order, quantity, extent) since they arise from real numbers and at the same time they are **sets** (intervals and/or distributions/membership functions whose graphs are indeed sets of real numbers). To be sure, the order on intervals are partial orders except for the case of real numbers, zero width intervals. Therefore, the solutions to (33) and (34) will possess a dual nature which are interval numbers and sets. Sets are the more general theoretical concept which includes the numerical (and not the other way around). Thus, our approach considers the solutions to (33) and (34) as sets.

Remark 12 *It is clear from (33) and (34) that the right and left sides of the relation are sets. That is, we can consider* x *to be a solution of (33) and (34) if* x *satisfies one of the following: 1)* $[A]x \cap [b] \neq \emptyset$*, 2)* $[A]x \subseteq [b]$*, 3)* $[A]x \supseteq [b]$*, 4)* $[A]x = [b]$ *or* $[A]x \leq [b]$*. Of course, 1) is the least constrained and 4) the most where equality in 4) means that* $[A]x \subseteq [b]$ *and* $[A]x \supseteq [b]$*. Therefore when strict equality is the interpretation of (33) and (34), interpretation 4), it is* ***set equality****.*

There are three different though related sets involved in solving (33) and (34).

1. The first are the parameter sets $[A], [b]$,
2. The second is the range set $[A]x$,
3. The third is the set of solutions $x \in \Omega$ which solves (33) and (34) according to one of the four interpretations given above defining what a solution to (33) and (34) means.

Based on our observations above, there are four types of solution sets Ω to (33) and (34)), one for each of the four interpretations ($[A]x \cap [b] \neq \emptyset$, $[A]x \subseteq [b]$, $[A]x \supseteq [b]$, $[A]x = [b]$ or $[A]x \leq [b]$) of what a solution to (33) and (34) means. The four types of solution sets are as follows (also see [49], [98] and [101]):

1. **Possible (united or optimistic) solution set**

   $$\Omega_{\exists\exists} = \{x \mid Ax = b, \exists A \in [A], \text{ and } \exists b \in [b]\}$$

   Note that if $\Omega_{\exists\exists} \neq \emptyset$ then $\{Ax \mid A \in [A], x \in \mathbb{R}^n\} \cap [b] \neq \emptyset$. In the context of fuzzy sets, $\Omega_{\exists\exists}$ is called the *optimistic set* by [49]. Some authors (see [98]) call $\Omega_{\exists\exists}$ the *united solution set* in the context of interval analysis. In [121], this is also called a *united solution set*. They go on to state that, "For fuzzy diagnosis, we may want to find all sets of symptoms x which can cause a set of diseases $b \in [b]$. This means that for any x we require only one $A \in [A]$ such that $Ax = b$ for some $b \in [b]$, which is essentially a united solution."

2. **Strongly (tolerable) possible solution "for all A, there is b" set**

$$\Omega_{\forall\exists} = \{x \mid Ax = b, \forall A \in [A], \text{ and } \exists b \in [b]\}$$

Note that if $\Omega_{\forall\exists} \neq \emptyset$, then $\{Ax \mid \forall A \in [A], x \in \mathbb{R}^n\} \subseteq [b]$. In other words, the range of all real matrix instantiations of $[A]$, $\{Ax$ where $A \in [A]\}$, are equals in some instantiation of $[b]$. In [121], this is called a *tolerable solution set*. They go on to state that, "For instance, in fuzzy control, we want to find all input states x which guarantee that the output states (an instantiation b) are within a tolerable range $[b]$ for all input-output relations ($A \in [A]$). This essentially the tolerable solution set case."

3. **Strongly (controllable) possible solution "there is A, for all b" set**

$$\Omega_{\exists\forall} = \{x \mid Ax = b, \exists A \in [A], \text{ and } \forall b \in [b]\}$$

Note that if $\Omega_{\exists\forall} \neq \emptyset$, then $\{Ax \mid A \in [A], x \in \mathbb{R}^n\} \supseteq [b]$. In other words, the range of all real matrix instantiations of $[A]$ contains all instantiations of $[b]$. In [121], this is called a *controllable solution set*. They go on to state that, "A question that naturally arises in fuzzy control is whether there exists some inputs x which can lead to any specified outputs $b \in [b]$ by appropriate choice (control) of the relation matrix $A \in [A]$. This is the controllable solution set."

4. **Necessary (pessimistic) solution set**

$$\Omega_{\forall\forall} = \{x \mid Ax = b, \forall A \in [A], \text{ and } \forall b \in [b]\}$$

Note that if $\Omega_{\forall\forall} \neq \emptyset$, then $\{Ax \mid A \in [A], x \in \mathbb{R}^n\} \subseteq [b]$ and $\{Ax \mid A \in [A], x \in \mathbb{R}^n\} \supseteq [b]$. This set is what [49] calls the *pessimistic set* and it signifies that all possible combinations of instantiations necessarily have a solution.

Remark 13 *It is emphasized that the sets $\Omega_{\exists\exists}, \Omega_{\forall\exists}, \Omega_{\exists\forall}, \Omega_{\forall\forall}$ need not be of the same type as the underlying uncertainty in the parameters. That is, when, for example, $\check{A}$, and $\check{b}$ are fuzzy intervals, the solution sets need not be fuzzy intervals. Or when we have an interval matrix $[A]$ and the interval vector right-hand side $[b], \Omega_{\exists\exists}$ can be star-shaped (non-convex) and thus not a box (interval) [29]. However, they will always have a random set representation ([116], [117]).*

Remark 14 *The interpretations given by Wang, Fang, and Nuttle, [121], are excellent. However, they seem completely unaware of the extensive work in the area of linear interval equations which began early in the decade of 1960-1969 soon after Moore's initial work was published. In particular they seem unaware of the early work of E. Hansen [31], [32] (chapter 4), J. Rohn [89], [90], [24], Arnold Neumaier [80], and S. P. Shary [98], [99], [100], [101], to name just four researcher.*

Remark 15 $\Omega_{\exists\exists}$ *is like plausibility/possibility while* $\Omega_{\forall\forall}$ *is like belief/necessity. Moreover,* $\Omega_{\forall\forall} \subseteq \Omega_{\exists\exists}$, $\Omega_{\forall\forall} \subseteq \Omega_{\forall\exists} \subseteq \Omega_{\exists\exists}$, *and* $\Omega_{\forall\forall} \subseteq \Omega_{\exists\forall} \subseteq \Omega_{\exists\exists}$ *.*

6.0.2 Key Issues

The study of systems of fuzzy linear systems will begin with interval linear systems. Recall that an interval $[a]$ is a fuzzy set with membership function

$$\mu_{[a]}(x) = \begin{cases} 1 \text{ for } \underline{a} \leq x \leq \overline{a} \\ 0 \text{ otherwise} \end{cases} .$$

Consider the most simple one-dimensional interval problem of solving

$$\begin{aligned} [a]x &= [b], \text{ in particular,} \\ [1,2]x &= [4,6]. \end{aligned} \tag{35}$$

Example 7 *Compute*
1) the ***possible solution set****,* $\Omega_{\exists\exists}$*, of the (35). That is, we seek*

$$\Omega_{\exists\exists} = \{x \mid ax = b, a \in [1,2], b \in [4,6]\} .$$

This results in

$$\Omega_{\exists\exists} = \{x \mid 2 \leq x \leq 6\} = [2,6]. \tag{36}$$

Compute
2) the ***strongly possible solution set "for all A, there is b"*** *:*

$$\Omega_{\forall\exists} = \{x \mid ax = b, \forall a \in [1,2], \exists b \in [4,6]\} .$$

This means that we seek solutions such that

$$\begin{aligned} [1,2]x &\in [4,6], \text{ that is,} \\ [1,2]x &\subseteq [4,6] \text{ so that} \\ \Omega_{\forall\exists} &= \{x \mid 4 \leq x \text{ and } 2x \leq 6\} = \emptyset. \end{aligned} \tag{37}$$

Another way to see this is to consider the following.

$$\begin{aligned}
1\cdot x &= \exists[4,6], \Rightarrow x\in[4,6] \textit{ and ... and}\\
\frac{5}{4}\cdot x &= \exists[4,6], \Rightarrow x\in[\frac{16}{5},\frac{24}{5}] \textit{ and ... and}\\
\frac{3}{2}\cdot x &= \exists[4,6], \Rightarrow x\in[\frac{8}{3},4] \textit{ and ... and}\\
\frac{7}{4}\cdot x &= \exists[4,6], \Rightarrow x\in[\frac{16}{7},\frac{24}{7}] \textit{ and ... and}\\
2\cdot x &= \exists[4,6], \Rightarrow x\in[2,3].
\end{aligned}$$

This set is empty $\Omega_{\forall\exists} = [4,6]\cap ...\cap[2,3] = \emptyset$. *In particular, to compute* $\Omega_{\forall\exists}$, *only the sets at the extremes (first and last intervals of our example) determine the resulting solution set.*
Next, compute the ***strongly possible solution set "there is A, for all b"****:*

$$\Omega_{\exists\forall} = \{x \mid ax = b, \exists a\in[1,2], \forall b\in[4,6]\}.$$

This means that we seek solutions such that

$$\begin{aligned}
[4,6] &\in [1,2]x, \textit{ that is,}\\
[1,2]x &\supseteq [4,6] \textit{ so that}\\
\Omega_{\exists\forall} &= \{x \mid 4\geq x \textit{ and } 2x \geq 6x\} = [3,4].
\end{aligned} \tag{38}$$

Another way to see this is to consider the following.

$$\begin{aligned}
\exists[1,2]x &= 4, \Rightarrow x\in[2,4] \textit{ and ... and}\\
\exists[1,2]x &= \frac{9}{2}, \Rightarrow x\in[\frac{9}{4},\frac{9}{2}] \textit{ and ... and}\\
\exists[1,2]x &= 5, \Rightarrow x\in[\frac{5}{2},5] \textit{ and ... and}\\
\exists[1,2]x &= \frac{11}{2}, \Rightarrow x\in[\frac{11}{4},\frac{11}{2}] \textit{ and ... and}\\
\exists[1,2]x &= 6, \Rightarrow x\in[3,6].
\end{aligned}$$

This is the set is $\Omega_{\exists\forall} = [2,4]\cap[3,6] = [3,4]$. *In particular, to compute* $\Omega_{\exists\forall}$, *only the sets at the extremes (first and last intervals of our example) determine the solution set.*
4) Calculate the ***necessary solution set****:,*

$$\begin{aligned}
\Omega_{\forall\forall} &= \{x \mid ax = b, \forall a\in[1,2], \forall b\in[4,6]\}\\
\Omega_{\forall\forall} &= \emptyset.
\end{aligned}$$

Example 8 *However, if our problem had been*

$$[1,2]x = [4,8]$$

then

$$\Omega_{\exists\forall} = \Omega_{\forall\exists} = \{4\}.$$

6.0.3 Constraint Interval Analysis Solution Approach

The CIA [50],[51] solution approach to the linear algebraic problem under uncertainty requires the associated definitions and operations which we give next. Let us now apply CIA to (35).

Example 9 *Solve*

$$[1,2]x = [4,6]$$

using CIA where we want to compute the four types of solution sets for this example. First, replace $[1,2]$ *by its CIA representation,*

$$[1,2] = \{z = (1-\lambda)1 + \lambda 2 = \lambda + 1, 0 \leq \lambda \leq 1\}.$$

Next, replace $[4,6]$ *by its CIA representation,*

$$[4,6] = \{z = (1-\gamma)4 + \gamma 6 = 2\gamma + 4, 0 \leq \gamma \leq 1\}$$

where we want the minimum and maximum of x *under the various interpretations of solutions. Our example problem* $[1,2]x = [4,6]$ *becomes*

$$\max / \min_{x} f(x, \lambda, \gamma) = (\lambda + 1)x = 2\gamma + 4, 0 \leq \lambda, \gamma \leq 1$$

Since we have a real-valued function and $\lambda + 1 \neq 0$,

$$x = \frac{2\gamma + 4}{\lambda + 1}$$

1. *For* $\Omega_{\exists\exists}$ *we want the minimum/maximum of* x *of*

$$x = \frac{2\gamma + 4}{\lambda + 1}, 0 \leq \lambda, \gamma \leq 1.$$

That is, we seek the minimum/maximum of x ***for some*** $\lambda \in [0,1]$, *and* ***for some*** $\gamma \in [0,1]$. *Clearly the minimum* $\underline{x}$ *will occur when* $\lambda = 1$ *and* $\gamma = 0$, *that is,*

$$2 \leq \gamma + 2 \leq \frac{2\gamma + 4}{2} \leq \underline{x}$$

Clearly, the maximum $\overline{x}$ *will occur when* $\lambda = 0$ *and* $\gamma = 1$. *that is,*

$$\overline{x} = \frac{2\gamma + 4}{\lambda + 1} \leq \frac{2\gamma + 4}{1} \leq 6.$$

This means that

$$\Omega_{\exists\exists} = [2, 6].$$

In other words, there exists $x \in [2, 6]$ *and there exists* $A \in [1, 2]$ *and there exists* $b \in [4, 6]$ *such that* $Ax = b$ *so that,*

$$\Omega_{\exists\exists} = \{x \mid 2 \leq x \leq 6\}.$$

The left endpoint of $\Omega_{\exists\exists}$ *is computed using* $\max \lambda = 1, \min \gamma = 0$ *and the right endpoint is computed using* $\min \lambda = 0, \max \gamma = 1$.
2. *For* $\Omega_{\exists\forall}$ *we want the minimum/maximum of* x *of*

$$x = \frac{2\gamma + 4}{\lambda + 1}, \forall 0 \leq \gamma \leq 1, \text{ for some } \lambda \in [0, 1].$$

For all γ *and some* $\lambda \in [0, 1]$, *it is clear that for the minimum* x, $\underline{x}$, *we need* $\lambda = 1$ *(maximum* λ *which makes the denominator the largest and the fraction smallest) and to hold for all* γ, $\gamma = 1$, *that is, both* λ *and* γ *are at their maximum. Thus,*

$$3 \leq \gamma + 2 \leq \frac{2\gamma + 4}{2} \leq \frac{2\gamma + 4}{\lambda + 1} = \underline{x}.$$

For the maximum $x, \overline{x}$, *we need* $\lambda = 0$ *(the minimum* λ *which makes the denominator the smallest and the fraction largest) and to hold for all* γ, $\gamma = 0$, *that is, both* λ *and* γ *are at their minimum. Thus,*

$$\overline{x} = \frac{2\gamma + 4}{\lambda + 1} \leq 2\gamma + 4 \leq 4$$

This means that the left endpoint is computed using $\max \lambda, \max \gamma$ *and the right endpoint using* $\min \lambda, \min \gamma$.

$$\Omega_{\exists\forall} = [3, 4].$$

3. *If we seek* $\Omega_{\forall\exists}$, *we want the minimum/maximum of* x *of*

$$x = \frac{2\gamma + 4}{\lambda + 1}, \forall 0 \leq \lambda \leq 1, \text{ for some } \gamma \in [0, 1].$$

For all λ *to hold, we need the minimum* x*,* $\underline{x}$*, to be hold even when* λ *is at its smallest value so that*

$$\underline{x} \geq \min_{0\leq\lambda\leq 1} \frac{2\gamma+4}{\lambda+1} = 2\gamma+4, \text{ for some } \gamma \in [0,1].$$

The least constraining case (smallest) will be when $\gamma = 0$ *so that*

$$\underline{x} \geq 2\gamma + 4 \geq 4.$$

For all λ *to hold, we need the maximum* $x, \overline{x}$*, to hold even when* λ *is at its largest value so that*

$$\overline{x} \leq \max_{0\leq\lambda\leq 1} \frac{2\gamma+4}{\lambda+1}, \text{ for some } \gamma \in [0,1].$$

Thus,

$$\overline{x} \leq \max_{0\leq\lambda\leq 1} \frac{2\gamma+4}{\lambda+1} = \gamma+2, \text{ for some } \gamma \in [0,1].$$

The least constraining case (smallest) will be when $\gamma = 1$ *so that*

$$\overline{x} \leq \gamma + 2 \leq 3.$$

This means that

$$\Omega_{\forall\exists} = \emptyset.$$

For $\Omega_{\forall\exists}$ *the left endpoint is computed using* $\min\lambda, \min\gamma$ *and the right endpoint using* $\max\lambda, \max\gamma$*.*

4. *The endpoints of* $\Omega_{\forall\forall}$ *are computed using for the left endpoint* $\min\lambda = 0 \max\gamma = 1$ *and for* $\max\lambda = 1, \min\gamma = 0$*. In particular*

$$\begin{aligned} \underline{x} &= \frac{2\gamma+4}{\lambda+1}, \lambda = 0, \gamma = 1 \\ &= 6 \end{aligned}$$

and

$$\begin{aligned} \overline{x} &= \frac{2\gamma+4}{\lambda+1}, \lambda = 1, \gamma = 0 \\ &= 2. \end{aligned}$$

Thus, $\Omega_{\forall\forall} = \emptyset$*.*

Example 10 *Had our example been*

$$[1,2]x = [4,8],$$

then

$$x = \frac{4\gamma+4}{\lambda+1}$$

and $\Omega_{\exists\forall} = \Omega_{\forall\exists}$ *since*

$$\underline{x} = \max_{\lambda}\max_{\gamma}\frac{4\gamma+4}{\lambda+1} = \min_{\lambda}\min_{\gamma}\frac{4\gamma+4}{\lambda+1} = \overline{x} = 4.$$

For the general simple interval problem

$$\begin{aligned} [a]x &= [b], 0 \notin [a] \\ [\underline{a},\overline{a}]x &= [\underline{b},\overline{b}], \end{aligned}$$

the CIA translation is,

$$x = \frac{width([b])\gamma + \underline{b}}{width([a]\lambda + \underline{a}}.$$

1. $\Omega_{\exists\exists}$ for $\underline{a} > 0$, is given by

   $$\Omega_{\exists\exists} = [\max\lambda = 1 \min\gamma = 0, \min\lambda = 0 \max\gamma = 1] = [\frac{\underline{b}}{width([a]\lambda + \underline{a}}, \frac{width([b])\gamma + \underline{b}}{\underline{a}}].$$

   For $\overline{a} < 0$ the analysis is similarly.

2. $\Omega_{\exists\forall}$ for $\underline{a} > 0$, is given by

   $$\Omega_{\exists\forall} = [\max\lambda = 1 \max\gamma = 1, \min\lambda = 0 \min\gamma = 0] = [\frac{width([b])\gamma + \underline{b}}{width([a]\lambda + \underline{a}}, \frac{\underline{b}}{\underline{a}}].$$

   For $\overline{a} < 0$ the analysis is similarly.

3. $\Omega_{\forall\exists}$ for $\underline{a} > 0$, is given by

   $$\Omega_{\forall\exists} = [\min\lambda = 0 \min\gamma = 0, \max\lambda = 1 \max\gamma = 1] = [\frac{\underline{b}}{\underline{a}}, \frac{width([b])\gamma + \underline{b}}{width([a]\lambda + \underline{a}}].$$

   For $\overline{a} < 0$ the analysis is similarly done.

4. $\Omega_{\forall\forall}$ for $\underline{a} > 0$, is given by

$$\begin{aligned}\Omega_{\forall\forall} &= [\min\lambda = 0 \max\gamma = 1, \max\lambda = 1 \min\gamma = 0] \\ &= [\frac{width([b])\gamma + \underline{b}}{\underline{a}}, \frac{\underline{b}}{width([a]\lambda + \underline{a}}].\end{aligned}$$

For $\overline{a} < 0$ the analysis is similarly done.

There is an alternate approach developed by J. Rohn, in Chapter 1 of [24] which is a summary from his earlier works. From the above and J. Rohn's research results, a solution, in principle can be obtained. Where we use these results in fuzzy/possibility optimization is to define a degree of feasibility $1-\alpha$, where α is the $\alpha-$level. Then,

$$A_{1-\alpha}x = b_{1-\alpha}$$

or

$$A_{1-\alpha}x \leq b_{1-\alpha}$$

is a linear interval system of equations or inequalities which can be solved as above.

7 Conclusion

Our exposition emphasized and delineated the differences between fuzzy set theory and possibility as it impacts optimization that use these entities. We indicate how the semantics are important in distinguishing which of the various approaches must be used in seeking a solution to associated models. The taxonomy of types of optimization that are most commonly linked to fuzzy sets and possibility distributions clarify where various approaches fit.

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