Introduction to Vector Fields
Calculus 3 – Section 14.1

• Definition: Vector Field in Two Dimensions
Let $f$ and $g$ be defined on a region $R \subset \mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns each point in $R$ a vector $(f(x, y), g(x, y))$. That is

$$\mathbf{F}(x, y) = (f(x, y), g(x, y))$$

A vector field $\mathbf{F} = (f, g)$ is continuous or differentiable on a region $R \subset \mathbb{R}^2$ if $f$ and $g$ are continuous or differentiable on $R$, respectively.

• Example 1: Let $\mathbf{F}(x, y) = (x, -y)$. Plot a sketch of the vector field on the domain $(x, y) \in [-2, 2] \times [-2, 2]$.

• Example 2: Let $\mathbf{F}(x, y) = (-y, x)$. Plot a sketch of the vector field on the domain $(x, y) \in [-2, 2] \times [-2, 2]$.

• Example 3: Let $\mathbf{F}(x, y) = (\sin(x), \sin(y))$. Plot a sketch of the vector field on the domain $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$. 

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**Definition:** Let \( \mathbf{r} = (x, y) \). A vector field of the form \( \mathbf{F}(x, y) = f(x, y)\mathbf{r} \), where \( f \) is a scalar-valued function, is a **radial vector field**. One particular type of radial vector field is 

\[
\mathbf{F}(x, y) = \frac{(x, y)}{|\mathbf{r}|^p}
\]

where \( p \) is a real number.

Notice that \( |\mathbf{F}| = 1/|\mathbf{r}|^{p-1} \) at each point except the origin.

**Example 4:** Sketch the vector field for \( \mathbf{F}(x, y) = -\mathbf{r}/|\mathbf{r}|^p \) with \( p = 3 \). (This is an example of the inverse square law for gravitational fields)

**Definition:** A vector field in \( \mathbb{R}^3 \) is a vector function of the form

\[
\mathbf{F}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))
\]

**Example 5:** Try to sketch the 3D vector field \( \mathbf{F}(x, y, z) = (x, y, z) \) on the domain \((x, y, z) \in [0, 1] \times [0, 1] \times [0, 1]\)
• **Definition:** Let \( \varphi \) be a differentiable scalar valued function (in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)), and define \( \mathbf{F} = \nabla \varphi \). This is called a **gradient field** or a **potential field**. The function \( \varphi \) is often called the **potential function**.

  * In \( \mathbb{R}^2 \), \( \mathbf{F} \) points in a direction that is ______________ to the level curves of \( \varphi \).
  * In \( \mathbb{R}^3 \), \( \mathbf{F} \) points in a direction that is ______________ to the ______________ of \( \varphi \).

• **Definition:** The level curves of a potential function are called **equipotential curves** (curves on which the potential is constant).

• **Definition:** A curve that is everywhere orthogonal to the equipotential curves is called a **streamline**

  * Streamlines are also called **flow lines** and follow the direction of the gradient field.

• **Example 6:** Plot several level curves, streamlines, and a potential field for the potential function \( \varphi = x^2 + y^2 \) on \([-2, 2] \times [-2, 2]\).

  ![Diagram](image)

• **Example 7:** Plot several level curves, streamlines, and a potential field for the potential function \( \varphi = \sin(x) \sin(y) \) on \([-\pi, \pi] \times [-\pi, \pi]\).

  ![Diagram](image)